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प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका

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PRE- PROCEEDINGS
of the
National Seminar
on
History of
Mathematical
Sciences
&
Applicable
Mathematics

March 7-9, 2003

Department of Mathematics and Statistics
Gurukula Kangri Vishwavidyalaya, Hardwar

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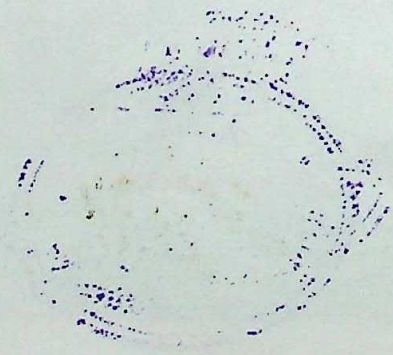

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सत्यमेव जयते

भारत के उप-राष्ट्रपति
VICE-PRESIDENT OF INDIA

MESSAGE

I am glad to know that the Gurukula Kangri Vishwavidyalaya, Haridwar is organising a National Seminar on "History of Mathematical Sciences and Applicable Mathematics" during March 7 – 9.

The Seminar offers welcome opportunity to the experts in the field of Mathematical Sciences and other related subjects to discuss the increasing relevance of Mathematics for the progress of human civilization and to evolve a strategy to face the challenges facing the discipline.

My best wishes for the success of the Seminar.

(BHAIRON SINGH SHEKHAWAT)

New Delhi
17th February 2003

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Message

Dear Shri Singh,

Pranam.

Please refer to your fax dated 20th February 2003 inviting Smt. Bhavnaben D. Chikhalia, Minister of State Tourism & Culture and Parliamentary Affairs, to inaugurate a National Seminar on "History of Mathematical Sciences and Applicable Mathematics" on 7 March, 2003. As the Parliament is in Session, Hon'ble Minister has expressed her inability to accept the invitation. However, she has asked me to convey her best wishes for the success of the Seminar.

With regards,

Yours sincerely,

(B.N. VAIDYA)
Private Secretary

Dr. S. L. Singh
Organising Secretary &
Principal, Science College
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D.O. No. 1-11/2003 (CM)

18th February, 2003

Dear Professor Kumar

This has reference to your letter No. 666-B of February 11, 2003. I am very happy to know that your University is organizing a National Seminar on History of Mathematical Sciences and Applicable Mathematics from March 7-9, 2003. I send my best wishes and greetings to all the participants and wish the seminar good luck and grand success.

With best wishes,

Yours sincerely,

(Arun Nigavekar)

Prof. Swantantra Kumar
Patron & Vice chancellor
National Seminar on History of Mathematical Sciences and Applicable Mathematics
Gurukula Kangri Vishwavidyalaya
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क्रमांक/*Ref. No.* 03-08 VC01
दिनांक/*Dated* 28-2-2003

Message

It is a matter of a great pleasure to know that the Vishwavidyalaya is organizing a National Seminar on the History of Mathematical Sciences and Applicable Mathematics from 7th to 9th March. I am sure that the seminar will highlight the achievements of ancient mathematicians of Indian and their contribution to the modern mathematics in its deliberations. The Vishwavidyalaya has been in the forefront for research activities in Vedic mathematics, and interaction in the seminar will provide a great opportunity to the scholars to interact with each other with a view to not only highlight the wisdom of ancient scholars but will also throw up new lines for the research in mathematics.

I wish the organizers and participants a great success in their venture.

Sada Nand

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MESSAGE

Dear Prof. Swatantra Kumar:

I am pleased to know that the Mathematics and Statistics Department is organizing a National Seminar which aims at discussing development of Mathematical Science and its applicable areas. Since antiquity, Mathematics enjoys top position among all Mundane Sciences. Vedic savants are best known for making significant fundamental contribution in Mathematics and Astronomy. Modern taste of Science, Technology and Civilization has become primarily Mathematical. Our Vishwavidyalaya is well known as a centre for the study of interaction Vedic and Modern Sciences. So the theme of this Seminar is very much in tune to the Gurukuliya tradition.

I extend a warm welcome to the peers and young delegates and wish a very successful deliberation.

Sudarshan

(Sudarshan Sharma)

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शुभकामना संदेश

परम हर्ष का विषय है कि गुरुकुल कांगड़ी विश्वविद्यालय के गणित विभाग में वैदिक गणित विद्या पर राष्ट्रीय विद्वत् सम्मेलन 7 मार्च 2003 से 9 मार्च 2003 तक आयोजित किया जा रहा है। जिस महा मनीषी अमर हुतात्मा स्वामी श्रद्धानन्द जी ने महर्षि दयानन्द सरस्वती के स्वप्न को साकार करने के लिए गुरुकुल की प्राक्तन शिक्षा पद्धति को संजीवन प्रदान किया उनके निर्देशानुसार ही गुरुकुल में वैदिक विद्याओं के पुनर्जागरण का कार्य निरन्तर प्रगति पर है। आज युग की पुकार है कि विश्व में अपना अस्तित्त्व बनाने तथा पूर्व मनीषियों की यशोगाथा को सत्य सिद्ध करने के लिए हमें वेदालोक में समस्त विद्याओं का उन्मेष करने पर बल देना होगा।

वैदिक परम्परा में गणित एवं ज्योतिष प्रायः पर्याय माने जाते हैं। ज्योतिष को वेदाङ्ग शास्त्रों में श्रेष्ठतम स्थान दिया गया है तथा ध्यान देने योग्य बात तो यह है कि ज्योतिष को चक्षु कहा गया है। जैसे कुछ भी देखने के लिए चक्षु का मानव जीवन में जो स्थान है वही वेद विद्या को जानने में ज्योतिष का स्थान है। ज्योतिष के आधार पर वर्ष, सम्वत्सर, मास, पक्ष, सप्ताह दिन, होरा निमेष आदि का कथन होता है। समग्र अन्तरिक्ष विद्या की जननी ज्योतिष ही है। इसी ज्योतिष वेदाङ्ग को तीन विभागों में विभक्त करके त्रिधा अंक, रेखा व बीज गणित के नाम से पुकारा जाता है। वेद में समस्त सत्य विद्याओं का बीज विद्यमान है; उस बीज को महान वृक्ष का रूप प्रदान करने में वेदाङ्ग जल सेचन का कार्य करते हैं। देश के प्राक् कालिक ऋषियों ने गहन तपश्चर्या करके वेद में विद्यमान गणित विद्या को विकसित किया है।

गुरुकुल कांगड़ी विश्वविद्यालय के मनीषी प्राध्यापक आज उसी दिशा में कार्य करते हुए इन सम्मेलनों के माध्यम से वेदों की ओर चलो का संदेश देकर वेदों की महत्ता पर प्रकाश डालेंगे। गणित विभाग के प्रोफेसर डॉ० एस०एल० सिंह, डॉ० वीरेन्द्र अरोड़ा, डॉ० वी०के० शर्मा, डॉ० महीपाल सिंह एवं डॉ० प्रभाकर प्रधान को मैं हृदय से साधुवाद देता हूँ कि ये सभी विद्वान् इस पावन ज्ञान यज्ञ के अनुष्ठान में पूर्ण निष्ठा के साथ लगे हैं। आज के युग में विद्वानों की, विशेषकर गुरुकुल के विद्वानों को जिस दिशा में कार्य करना चाहिए हमारे विद्वान् उसी दिशा में निःसन्देह प्रशंसनीय कार्य कर रहे हैं। उक्त सम्मेलन में विभिन्न विश्वविद्यालयों तथा शोध पीठों से जो विद्वान् इस सारस्वत यज्ञ में अपनी पावन ज्ञान की आहूति देकर सुविचार की सुगन्ध उत्पन्न करेंगे, उन सभी का हार्दिक स्वागत करते हुए सम्मेलन की पूर्ण सफलता की मङ्गल कामना करता हूँ।

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वेदेषु गणित विद्या बीजानि

वेद प्रकाशः शास्त्री

परमपिता परमात्मना या वेदवाणी जगत् कल्याणाय मानवसृष्टेरादौ चतुर्णामृषीणां हृदयगुहायां प्रकाशिता सैव वेद वाणी सर्वज्ञानमयीति निगद्यते। सर्वाः सत्यविद्या तत एवं प्रादुर्भवन्ति। यथा विविधानां विधानां बीजानि वेदे विलोक्यन्ते तथैव गणित विद्या बीजान्यपि प्रमासमानानि सन्ति। वेदे या छान्दसी रचना सा गणित विद्योद् गायिका। वर्णसंख्या माश्रित्य छन्दस ऊर्मयः संगीत माधुरी मुल्लासयन्ति। गणक पदं गणित विधां प्रबोध यति। यथा –

गणानां त्वा गणपतिं हवामहे। १ ऋक् २, २३, १
गणानां त्वा गणपतिं हवामहे निधिनां त्वा
निधिपतिं हवामहे। यजु० २३, १६ गण्या – १ ऋक् ३/७/५
व्रातं व्रातं गणं गणं १७ ऋक् ३/२६,६/५,५३,११

वेद संख्यावाचक शब्द प्रयोगः संख्या प्रतिपत्तये जिज्ञासून प्रबोधयति। कञ्चित् समसंख्या वाचकाः शब्दाः सन्ति कञ्चिच्च विषम संख्या वाचका शब्दा विलोक्यन्तो यथा विषम संख्या समुच्यते –

एका च में तिस्रश्च में तिस्रश्च में पञ्चच में पञ्चच में सप्तच में सप्तच में नवच में नवच में एकादश च में एकादशच में त्रयोदश च में त्रयोदश च मे पञ्चदश च में पञ्चदश च में.....

यजुर्वेद १८, २१६

समसंख्या वाचकाः शब्दा अपि प्राप्यन्ते – यथा – चतस्रश्च मेऽष्टौ च मेऽष्टौ च में द्वादश च में द्वादश च में षोडश च में षोडश च में विंशतिश्च मे विंशतिश्च में चतुर्विंशतिश्च मे चतुर्विंशतिश्च में..... यजुर्वेद १८, २५

क्रमशः संख्या परिगणनयामपि वेदे विलोक्यते यथा

सविता प्रथमेऽहन्थग्निर्द्वितीये वायुस्तृतीये आदित्यश्चतुर्थे चन्द्रमाः पञ्चमेऽऋतुः षष्ठे मरुतः सप्तमे बृहस्पतिरष्टमे मित्रो नवये बरुणो दशम इन्द्र एकादशे विश्वदेवा द्वादशे।।

यजुर्वेद ३६, ६

शून्य विषये वेदेषु पदं विद्यते तन्नामस्ति खम्। खं पदेन भवति शून्यावबोधः – यथा

ओ३म् खं ब्रहम। यजु० ४०, १७
अंग्धि खम्। ऋक् १०, १५६, ३,
खे रथस्य। ऋक् ८, ६१, ७
कः सप्त खानि विततर्द शीर्षाणि। अथर्व १०, २, ६
विषाहि गृणते खम्।
ऋक् ४, ११, २

बीज गणित विषये सांकेतिकी दृष्टिरपि तत्रान्वेष्टुं शक्यते यथा खं पदेन शून्यं ज्ञायते ब्रहम

आचार्य उपकुलपतिश्च, गुरुकुल कांगड़ी विश्वविद्यालयस्य

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पदेन एकस्यबोधो भवति काल पदेन त्रयस्य ज्ञानं क्रियते युग पदेन चतुर्गृध्यते। एवं प्रकारेण परवर्तिमिः सूरिमिर्नन्नु एष संकेत क्रमः परिवर्धितः। बहुषु स्थलेषु एवं विद्याः प्रयोगा द्रष्टुं शक्यन्ते।

ज्यामिति विषये खलु तत्र
ऋग्वेदे स्पष्टतया प्राप्यते
मन्त्रो यत्र रेखागणितावबोधो भवति। यज्ञ वेदिमभि
लक्ष्य प्रश्नमालिका या विधते
सा रेखागणित प्रकाशिका।
यथा–कासीत् प्रमा प्रतिमा कि निंदानम्
आज्यं किमासीत् परिधिः क आसीत्।
छन्छः किमासीत् प्र उ गं किमुक्यं
यद् देवा देवमयजन्तविश्वे।। ऋक् १०, १३०, ३

अस्मिन् मन्त्रे प्रमा, प्रतिमा, निदानम्, परिधि, छन्द, प्र उ ग, पदानि रेखागणित बीजं व्यक्तरूपेण प्रदर्शयन्ति। एवमेव वृत्त विषये एको मन्त्रः
प्राप्यते–यथा
चतुर्भिः साकं नवतिं च नामामिः।
चक्रं न वृत्तं व्यतींखीविपत्।।
एवमेव वर्ष चक्र वर्णन माध्यमेन प्राप्ताः
शब्दा सन्ति–यथा–
द्वादश प्रधयश्चक्रमेकं त्रीणि क उ तच्चिकेत।
तस्मिन् साकं त्रिशता न शंकवो
अर्पिता षष्टिर्न चलाचलासः।।

ऋक् ११६, १७ १७८

विदुषां कर्त्तव्यमिदं विधते सम्प्रति यत्ते वेदे निस्तन्द्रा भूत्वा निगूढं तत्वजातं सात्विकधिया चिन्तयन्तु। हर्षस्यायं विषयो विधते गुरुकुल कांगड़ी विश्वविद्यालये गणित विभागे राष्ट्रीयं विद्वत् सम्मेलनं वैदिक परिप्रेक्ष्ये समायोजकैरायोज्यते। एतदर्थं प्रो एस.एल. सिंह प्रो. वीरेन्द्र अरोड़ा प्रो० वी०के० शर्मा प्रो. महीपाल, डॉ० पी. प्रधान पदवाच्याः सर्वे धन्यवादार्हाः सन्ति। सम्मेलनस्य सार्वत्रिकं गौरवं कामये।

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डॉ० महावीर अग्रवाल
प्रोफेसर संस्कृत एवं कुलसचिव

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संदेश

बृहत्संहिता में वास्तु विज्ञान

गुरुकुल कांगड़ी विश्वविद्यालय का यह सौभाग्य है कि गणित विभाग द्वारा आयोजित राष्ट्रीय शोध संगोष्ठी में देश के मूर्धन्य विद्वान् उपस्थित होने जा रहे हैं। एक मास से भी न्यून अवधि में गणित के लब्ध प्रतिष्ठ विद्वान् प्रो० एस०एल० सिंह, प्राचार्य विज्ञान महाविद्यालय तथा प्रो० वीरेन्द्र अरोड़ा, संकायाध्यक्ष ने देश के वयोवृद्ध, ज्ञानवृद्ध विद्वानों को इस आयोजन में निमन्त्रित किया है और ये वरेण्य विद्वान् यहां उपस्थित हैं।

गणित को अनेक विद्याओं का मूल कहा जाता है। संसार का कोई व्यक्ति गणित विद्या से अछूता नहीं है। विज्ञान्, कम्प्यूटर, अभियान्त्रिकी, प्रौद्योगिकी, वाणिज्य, अर्थशास्त्र आदि का तो यह प्राण है। संपूर्ण संसार को गणित विद्या की शिक्षा देने वाले भारतीय आचार्यों ने ही शून्य की गवेषणा की थी। ज्योतिष वेदांग के अंग रूप में इस विद्या का विस्तार यहां होता रहा है। मैं शोध संगोष्ठी की पूर्ण सफलता की मंगल कामना करते हुए तथा विशेष रूप से प्रो० एस०एल० सिंह जी के प्रति हार्दिक सम्मान प्रकट करते हुए वास्तुशास्त्र पर कुछ विचार बृहदसंहिता के आधार पर प्रस्तुत कर रहा हूँ।

भवन निर्माण कला सौन्दर्य से जुड़ा हुआ विज्ञान एवं कला दोनों है, जिसका काम व्यवस्थित एवं सुसज्जित भवनों के माध्यम से भावनाओं को व्यक्त एवं जागृत करना है।
समराङ्गण सूत्रधार के रचयिता भोजने स्थापत्य को इस प्रकार परिभाषित किया है।

शास्त्रं – कर्म तथा प्रज्ञां शीलं च क्रियया न्वितम्।

वास्तु शब्द 'वस् धातु' से निर्मित है जिसका अर्थ यास्क के निरूक्त में "रहना" है। पुरातत्वाविदों ने सिन्धु घाटी सभ्यता के समय के बहुत से स्मारक एवं नगर नियोजन उत्खनन में पाये हैं। ऋग्वेद एवं अन्य वैदिक ग्रंथों में भी भवन निर्माण से जुड़े हुए महत्वपूर्ण शब्द मिलते है, यथा – वास्तु, वास्तोष्पति, अर्म, अर्मक, गय, हर्म्य, पस्त्य, ओकः, गृहं, दम आदि। संस्कृत में ऐसे अनेक ग्रंथ हैं जोकि भवन निर्माण, नगर नियोजन, सहायक अभियांत्रिकी एवं शिल्प के मुख्य आयामों को बताते हैं। उनमें से मुख्य है – अग्नि मत्स्य संहिता, कलिकागम, गर्गसंहिता, नारद संहिता, प्रसाद मंडन वास्तुशास्त्र, मानसार, समरांगणसूत्रधार, विश्वकर्मा प्रकाश आदि।

इन ग्रन्थों में वराह मिहिर की बृहत् संहिता ज्योतिष, खगोल शास्त्र, मणि विज्ञान, धातु विज्ञान, कृषि भवन निर्माण कला, अर्थशास्त्र का विश्वज्ञान कोष है। वराह मिहिर (505 ई0) के सम्बंध में कहा जाता है कि ये उज्जैन के निकट रहने वाले एक विशेष प्रतिभा के धनी वैज्ञानिक

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थे जो कि प्राचीन परम्पराओं एवं ऋषि मुनियों में विश्वास रखते हुए विषय के अध्ययन में वैज्ञानिक पद्धति अपनाते थे। उत्पल के अनुसार वराह की निम्न रचनायें हैं। (1) पंचसिद्धांतिका (2) जातक (3) यक्षेस्व – मेधीय यात्रा (4) योगमाला (5) विवाह पतल (6) बृहद् संहिता। आगम शास्त्र एक विशेष विधा है जिसमें भवन निर्माण, सामुद्रिकी, आयुर्वेद, रत्न विज्ञान आदि पर विद्वत्ता पूर्ण चर्चा की गई है। वराह ने अध्याय 53 में भवन निर्माण के कुछ अत्यन्त महत्वपूर्ण बिन्दुओं पर प्रकाश डाला है। जोकि इस प्रकार हैं –

(1) भवन निर्माण के लिए भूमि का चयन – किसी भी प्रकार के भवन के लिए भूमि चयन की विशेष महत्ता है। वराह ने कुछ ऐसे तथ्यों का वर्णन किया है जो कि स्थायी निर्माण के लिए सहयोगी हो सकते हैं जिसके अनुसार भूमि कोमल, सुगन्धित व मधुर स्वाद वाली होनी चाहिए। नीचे से खोखली नहीं होनी चाहिए और लाभदायक औषधियों वनस्पतियों से भरी होनी चाहिए।

शस्त्रौषधिद्रुमलता मधुरा सुगन्धा।
स्निग्धा, समा न सुषिरा च मही नराणाम्।। (बृ. सं, 88)

मिट्टी जो कि शुभ्र, लाल पीली, और काली होती है, विभिन्न वर्ग के लोगों के लिए लाभदायक होती है। किन्तु आवश्यक यह है कि यह विभिन्न प्रकार की घासों से ढकी होनी चाहिए – दर्भा, सार, दूब और कांस। लेकिन मुख्य सिद्धांत है आत्म सन्तुष्टि –

तत्तस्य भवति शुभदं यस्य च यस्मिन् मनो रमते। बृ.सं. 64

वही स्थान शुभ है जहाँ मन प्रसन्नता अनुभव करे। संक्षेप में भवन बनाने के लिए वहीं भूमि उपयुक्त है जो कि समतल हो, उपजाऊ और कोमल हो, जिससे कि भवन बिना बाधा के बनाया जा सके।

(2) विभिन्न प्रकार के निवास करने वाले घरों की विभाऐ – वराह ने राजा या धनी व्यक्तियों के लिए तथा दैवज्ञ पुरोहित, वैद्य आदि विभिन्न प्रकार के नागरिकों के लिए घरों की माप का वर्णन किया है –

उत्तममष्टाभ्यधिकं हस्तशतं नृपगृहं पृथुत्वेन।
अष्टाष्टोनान्येव पञ्च सपादानि दैर्ध्येण।। बृ.सं. 54

राजा का महल पाँचों में से एक विशेष प्रकार का होता है उनमें सबसे उत्तम चौड़ाई में होते हैं और अन्यों में 50, 46, 42, 54 मीटर और 38 मीटर क्रमशः होते हैं। उनकी लम्बाई चौड़ाई से एक चौथाई अधिक होती है।

इसी प्रकार सेनापति, युवराज महामंत्री और दूसरे अधिकारियों के घरों के माप का वर्णन 5 से 9 वें पदों में मिलता है। यहाँ आगे कहा गया है कि दैवज्ञ, पुरोहित और वैद्य के गृहों में चौड़ाई में 20, 18 मीटर और 16 मीटर, उनकी चौड़ाई से 1/6 वाँ हिस्सा अधिक होनी चाहिए।

वराह के अनुसार सामान्यतः एक भवन की लम्बाई और चौड़ाई नाप में बराबर होनी चाहिए। केवल उन घरों को छोड़कर जहाँ पर एक बड़ा हाल होता है। वहाँ लम्बाई, चौड़ाई से दो गुनी होनी चाहिए।

वास्तुनि यो विस्तारः स एव चोच्छ्राय निश्चयः शुभदः।
शालैकेषु गृहेष्वपि विस्तारादद्विगुणित – दैर्घ्यम्।। बृ.सं.।।

पशुओं के लिए शस्त्रागार के लिए यज्ञशाला एवं खेल आदि के लिए कोई विशेष नाप नहीं दी गई है। वराह ने हालांकि यह अनुभव किया कि विद्वान् वास्तु शास्त्रों के अनुसार रहने वाले स्थानों की लम्बाई 50 मीटर से अधिक नहीं होनी चाहिए।

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हस्तोऽनेच्छन्ति शास्त्रकारा हस्तशतादुच्छ्रित परतः। बृ.सं.16

वराह ने ऊपरी तलों के नाप के विषय में भी लिखा है।

विस्तारषोडशांशः सचतुर्हस्तो भवेद् गृहोच्छ्रायः।
द्वादशभागेनोनो भूमौ भूमौ समस्तानाम्।। बृ.सं. 22

प्रथम तल की ऊंचाई 1/16 वाँ हिस्सा चौड़ाई से कम होनी चाहिए। उससे ऊपर के तलों की ऊंचाई निचले तल से 1/12 वाँ भाग कम होनी चाहिए।

वराह के अनुसार घर की दीवारें पकी हुई ईंटों की तथा कमरों की चौड़ाई के 1/16 वाँ भाग होनी चाहिए। लेकिन लकड़ी से बने हुए घरों पर कोई प्रतिबन्ध नहीं था।

व्यासात् षोडशभागः सर्वेषां सद्मनां भवति भित्तिः।
पक्वेष्टकाकृतानां दासकृतानां तु न विकल्पः।। बृ.सं. 23

दरवाजों, खम्भों आदि की नाप भी यहाँ दी हुई है। (24–36)

राजा अथवा धनी और किसी विद्वान् का घर जिसके चारों ओर बरामदा होता है और चार दरवाजे होते है 'सर्वतोभद्र' कहलाता है।

अप्रतिषिद्धालिन्दं समन्ततो वास्तु सर्वतोभद्रम्।
नृपविबुधसमूहानां कार्यं द्वारैश्चतुर्भिरपि।। बृ.सं. 39

वराह ने निवास स्थानों का नामकरण दीर्घप्रकोष्ठों और कक्षों की स्थिति के आधार पर किया है। तीन कक्षों के घर को जिसमें उत्तरी प्रकोष्ठ नहीं है को 'हिरण्यनाभ' कहा गया है। इसी प्रकार 'सुक्षेत्र' में पूर्वी प्रकोष्ठ नहीं होता है। दक्षिण प्रकोष्ठ से रहित घर को कली और पश्चिम हाल से रहित घर को पाकसघना कहते हैं।

दो दीर्घ प्रकोष्ठों घर जिसमें केवल पश्चिमी और दक्षिणी प्रकोष्ठ होते हैं, 'सिद्धार्थ' के नाम से जाना जाता है। जिसमें पश्चिमी और उत्तरी प्रकोष्ठ होते हैं, यमसूर्या; जिसमें उत्तरी और पूर्वी प्रकोष्ठ होते है, दण्ड; जिसमें पूर्वी और दक्षिणी प्रकोष्ठ होते है, वट; जिसमें पूर्वी और पश्चिमी प्रकोष्ठ होते है, गृहचुल्ली और जिसमें दक्षिणी और उत्तरी प्रकोष्ठ होते हैं उसे 'काच' कहते हैं।

51–69 मंत्रों में वास्तुपुरूष एवं उसके सहयोगी देवताओं का वर्णन है। वराह ने रसोई भण्डार घर आदि की दिशाओं का भी वर्णन किया है।

ऐशान्यां देवगृहं महानसं चापि कार्यमाग्नेय्याम्।
नैर्ऋत्यां भाण्डोपस्कमरोऽर्थधान्यानि मारुत्याम्।। बृ.सं. 118

घर में पूजा स्थल उत्तर पूर्व में, रसोईघर दक्षिण पूर्व में, भण्डार घर दक्षिण पश्चिम में और कोषागार एवं अन्नागार उत्तर पश्चिम में होना चाहिए।

वराह के अनुसार – विशेष प्रकार के वृक्ष घर कें चारों ओर लगाए जाने चाहिए। घर के चारों ओर कांटेदार वृक्ष जैसे खदिर, दूधी – वृक्ष जैसे अर्क और फलदार बड़े वृक्ष जैसे कि आम आदि नहीं लगाने चाहिए। इसका कारण यह है कि इनसे दुर्घटना हो सकती है। समीप रहने वाले लोगों को भय हो सकता है। ये वृक्ष भवनों को हानि भी पहुँचा सकते हैं।

वराह ने परामर्श दिया है कि यदि उपरोक्त वृक्ष घर के आसपास है तो उन्हें उखाड़ना नहीं चाहिए अपितु कुछ सुंदर व उपयोगी वृक्ष उनके मध्य में रोपित कर देने चाहिए। ऐसे अच्छे वृक्षों में निम्ब (नीम), वकुल, अशोक आदि है।

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ऐसे वृक्ष भवन की सुंदरता को तो बढ़ाते ही है साथ ही मानव जीवन को भी स्वस्थ व सुखी रखने में सहयोग प्रदान करते हैं। ऐसा विचार आज भी पर्यावरण की दृष्टि से अत्यंत महत्वपूर्ण है। अगला प्रश्न है कि हमारा पड़ोसी कौन हो? सचिव और धूर्त व्यक्ति बिल्कुल नहीं। ऐसे लोग धन छीन सधकते है या बच्चों के जीवन को भय उत्पन्न कर सकते है। घर चैत्य के निकट नहीं होना चाहिए। जिसके अनेक अर्थ हैं– पत्थरों का ढेर, श्मशान भूमि, बौद्ध या जैन मंदिर या गूलर का वृक्ष। इसी प्रकार घर के चारों ओर वाल्मीक (बांबी) और गढ्डा नहीं होना चाहिए क्योंकि ये निवासियों के जीवन के लिए खतरा बन सकते है।

बहुत से कारण हैं जोकि निवास स्थान की सुंदरता और पवित्रता में सहयोग प्रदान करते हैं।

वराह के अनुसार – मुख्यद्वार प्रत्यक्ष रूप से विशाल और घर के अन्य सभी द्वारों से अलग होना चाहिए। मुख्यद्वार किसी भी दूसरे द्वार से अलग होना चाहिए तथा किसी भी अन्य द्वार से निर्माण एवं सुन्दरता में छोटा नहीं होना चाहिए। यह द्वार पवित्र वस्तुओं जैसे घड़ा, फलों का गुच्छा आदि से परिपूर्ण होना चाहिए। इस प्रकार के भव्य मुख्य द्वार से जो भी व्यक्ति अंदर आता है वह प्रवेश करते समय प्रसन्नता अनुभव करता है।

वराह के अनुसार पूरा घर विभिन्न प्रकार के फूलों, सुंदर आर्क (अर्धवृत्त) सुसज्जित जलपरिपूर्ण कलशों से शोभायमान होना चाहिए। घर को विभिन्न इत्रों से सुगन्धित रखना चाहिए। घर को पवित्र वैदिक मंत्रों के पाठ से धार्मिक बनाकर रखना चाहिए।

भूरिपुष्पविकरं सतोरणं त्तोयपूर्ण किणशोपशोभितम्।
धूपगन्ध बलिपूजितामरं ब्राह्मणध्वनियुतं विशेद्गृहम्।। (बृ.सं. 25)

उपर्युक्त कारक न केवल घर को सुशोभित करते है अपितु उस घर में रहने वालों की प्रसन्नता और विश्राम में भी सहयोग करते है। आज भी आंतरिक सज्जा भवन निर्माण की एक प्रमुख शाखा है।

इस विषय में वराह के परामर्श व्यावसायिकों के लिए बहुत उपयोगी है, विशेषतः गृहणियों के लिए सदैव उपयोगी रहेंगे।

इस प्रकार वराह भवन निर्माण कला के विभिन्न पक्षों को प्रकट करते हुए यह बताते हैं कि भारत ने 1500 वर्ष पूर्व किस प्रकार सामाजिक एवं सांस्कृतिक प्रगति की थी? सभ्यता का उच्च स्तर एवं शाही रहन–सहन इस ग्रंथ में झलकता है।

वराह अपनी स्पष्ट, पारदर्शी एवं आनन्ददायक भाषा शैली से भवन निर्माण विज्ञान के विभिन्न पहलुओं का वर्णन करने में सफल हुए।

भवन निर्माण के साथ सही प्रकार के पौधे के रोपण को भी महत्व दिया गया। यह परामर्श आज भी उपयोगी है जब हम वनों के कटान और पर्यावरण असंतुलन की बात करते हैं। वस्तुतः भवन निर्माण का अध्ययन ज्ञानवर्धक एवं प्रेरणादायक दोनों है।

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मदन कौशिक
MADAN KAUSHIK
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Member of Legislative Assembly, Uttaranchal

उत्तरांचल शासन

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दिनांक 26-02-2003

Message

This is a matter of great pleasure that the National Seminar on History of Mathematical Sciences and Applicable Mathematics is being organized by the Department of Mathematics and Statistics, Gurukula Kangri University, Hardwar form 7th to 9th March, 2003. There should be a good interaction and exchange of ideas between mathematicians and traditional scholars not only to extend the frontiers of mathematics but also to discover ways and means of harnessing the forces of knowledge for the service of human race and for betterment of quality of human life.

I extend my best wishes for the success of the Seminar.

Madan Kaushik

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FROM ORGANIZING SECRETARY - S. L. Singh

This is an overwhelming occasion that we are organising a National Seminar on History of Mathematical Sciences and Applicable Mathematics. We could get only 28 days to organize this Seminar. First I contacted Professor G. S. Panday of Ujjian and Professor B. S. Yadav of Dehli, and they very kindly not only agreed to deliver lectures but also extended their full cooperation for organizing this Seminar at a very short notice. Our Vice-Chancellor Professor Swatantra Kumar has shown special interest and enthusiasm in this Seminar. Indeed, we are able to organize this seminar because of his dynamic leadership and valuable co-operation from the administration, viz, Professor Ved Prakash, Pro-VC, Professor Mahavir Agrawal, Registrar and Shri Jai Singh Gupta, Finance Officer.

On behalf of the organizing committee and my own behalf, I extend a very warm welcome to our Chief guest, distinguished guests, & scholars, delegates, friends, colleagues, brothers and sisters.

We organized national level symposia on Mathematics and Vedic Mathematics in 1989 & 1995; the 64th Indian Mathematical Society Conference and Symposia on Fixed Point Theory & Vedic Mathematics were also organized in 1998. This was followed by an International Conferences on History of Mathematics in 1999 which was organized under the chairmanship of Professor B.S. Yadav. All these events were very well received by scholars and Mathematicians. Perhaps, this is the last seminar that I am organizing with very active cooperation of my colleagues, with the risk of missing many names; I must mention Virendra Arora, V. K. Sharma, M. P. Singh, Prabhakar Pradhan, D. D. Sharma, R. D. Kaushik, Rajendra Kumar Agrawal, P. P. Pathak, L. P. Purohit, S. C. Dhamija, R. D. Singh, Shri Krishna and Karmjeet Bhatia. Once again, I take the risk of missing a few more names of my students and non teaching staff and mention the untiring

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work of Ajendra Kumar, Anurag Naithani, Ashish Kumar, Hansraj Joshi, Yashpal Rana, M.G. Upadhyaya and Prakash Tiwari.

I take this opportunity to extend my indebtedeness and sincere thanks to Dr. Raj Kumar Rawat (G. K. Pharmacy), Mr. K. L. Kashyap (Mandi Govind Garh Charitable Trust Dharmshala, Hardwar), Mr. Raj Kumar Mahajan (Mahajan Bhawan) and Mr. T. P. Kumaria (GKV's consultant architect), Mr. Rajendra Tiwari (basically Mathematician and Asst. Transport Commissioner, Uttaranchal Pradesh), his colleague Mr. M. M. Jaiswal, and Mr. Chandra Kiran Saini. However, cooperation alone can not bring happiness unless there is a good support involving finances. I take this opportunity to express indebtedeness to His Holiness Swami Chidanandji (Chief President, Parmath Niketan, Rishikesh), Rajrani Trust (Jalandhar), Shri Madan Kaushik (M.L.A., Hardwar), Shri M. C. Garg (Periodica India, Faridabad), M/s Oriental Assurance Co. (Hardwar), Trustees of Mandi Govind Garh Charitable Trust Dharmashala, Hardwar, General Manager, Gurukula Kangri Pharmacy, Hardwar and Furniture House (Hardwar).

Finally, I extend a very warm welcome to all friends and delegates, and hope that their active participation alone can bring a great success to the Seminar.

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Harbanshlal Sharma
(02.02.1920 - 02.12.2002)

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दानवीर - हरबंस लाल शर्मा

वीरेन्द्र अरोड़ा

गुरूकुल कांगड़ी विश्वविद्यालय के पूर्व कुलाधिपति, आर्य प्रतिनिधि सभा पंजाब के प्रधान पं० हरवंशलाल जी का जन्म 2 फरवरी, 1920 में जालन्धर जनपद के रूड़की कलां ग्राम में एक पौराणिक ब्राह्मण परिवार में हुआ था। आपके पिता श्री कर्मचन्द शर्मा और माता श्रीमती लालदेवी बहुत ही धार्मिक विचारों के थे। करांची में रहते हुए ही पं० कर्मचन्द जी अपने भाईयों के साथ आर्य समाज से जुड़ गये थे। सभी भाई आर्य समाज की विचारधारा में ऐसे रंग गये कि आर्य समाज के दीवाने हो गये। माता-पिता का प्रभाव बालक हरवंशलाल पर बहुत गहरा पड़ा। ज्येष्ठ पुत्र होने के नाते इन्होंने आर्य समाज तथा साधु-विद्वानों की सेवा का कार्य बाल्यावस्था से ही प्रारम्भ कर दिया।

कुछ वर्ष करांची में नौकरी के पश्चात अपने एयरफोर्स में 10 वर्ष तक सेवा की और फिर नौकरी को तिलांजली देकर साईकिल पार्ट्स का कार्य प्रारम्भ कर दिया। ईमानदारी, परिश्रम, लगन एवं ईश्वर विश्वास आदि गुणों से आपका उद्योग दिन-दूनी रात चौगुनी प्रगति करता रहा। इनकी धर्मपत्नी श्रीमती राजकुमारी भी बहुत धार्मिक विचारों की हैं।

2 दिसम्बर 2002 को अपनी जीवन यात्रा पूर्ण कर पंडित जी अनन्त में विलीन हो गये। पंजाब के इस भामाशाह की मृत्यु से जहां कई सामाजिक, धार्मिक एवं शैक्षणिक संस्थायें अनाथ महसूस करने लगीं, वहीं आर्य जगत ने भी एक अमूल्य हीरा खो दिया।

मैं पंडित हरवंश लाल शर्मा जी से सन् 1965 से परिचित हूँ, जब मैंने गुरुकुल में शिक्षण कार्य प्रारम्भ किया था। मुझे उनके निकट आने का भी सौभाग्य प्राप्त हुआ। पंडित जी एक सरल, सच्चे आर्य-समाजी, धार्मिक एवं दयावान व्यक्ति थे। वे सदैव नम्र रहते थे। पंडित जी कर्मठ, कर्तव्य परायण निष्ठावान व दानवीर व्यक्ति थे। गुरूकुल के प्रति उन्हें विशेष प्रेम था। वे गुरूकुल के लिये प्रत्येक प्रकार की सेवा के लिये तत्पर रहते थे।

गीता में श्री कृष्ण ने यज्ञ, दान और तप को नित्य करणीय कर्म बताया है। यज्ञ से देवताओं की तृप्ति होती है। नियत समय पर वर्षा और अन्न उत्पन्न होते हैं। दान से प्राणिमात्र के जीवन की रक्षा होती है तथा तप से व्यक्ति संयमी, परोपकारी और सेवा भावी

संकायाध्यक्ष, विज्ञान एवं प्रौद्योगिकी संकाय, गुरुकुल कांगड़ी विश्वविद्यालय, हरिद्वार
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हो जाता है। व्यक्ति को समाज और राष्ट्र से जोड़ने में यज्ञ, दान और तप की मुख्य भूमिका बताई गई है। पंडित हरबंश लाल शर्मा जी में तीनों गुण विद्यमान थे। उन्हें धनी होने का लेश मात्र भी अभिमान नहीं था। नित्य यज्ञ और दान करना उनके जीवन के अभिन्न अंग बन गये थे। आर्य समाज की विभिन्न संस्थाओं को इस कार्य के लिये वह पर्याप्त राशि सात्त्विक भाव से प्रदान करते थे। उनके हृदय में कभी प्रतिदान की भावना पैदा नहीं हुई। उन्होंने निष्काम भाव से पुण्य-कार्य किए। इसलिए उनके दान को सात्त्विक कहा जाना चाहिए। अपने जीवन में उन्होंने इसी भाव से प्रत्येक व्यक्ति की सहायता की।

गुरूकुल कांगड़ी विश्वविद्यालय और स्वामी श्रद्धानन्द उनके आदर्श रहे हैं। उनकी मान-मर्यादा के लिए पंडित जी सदैव तत्पर रहते थे। वह निर्भीक, निर्लोभ तथा आत्म-विश्वास से पूर्ण व्यक्त्वि के धनी थे। वह अत्यन्त विनम्रता के साथ अपनी बात कह देते थे पर उनका दृष्टिकोण कठोर और हितकारी होता था। पंडित जी ने सदैव धर्म तथा नीति की बात कही तथा सच्चे मन से जो कहा वही किया। गुरूकुल की प्रबन्ध नीति में उनकी यही दृष्टि थी।

पंडित जी के आदर्श एवं भावालोक में उनके द्वारा सत्यापित ''राजरानी ट्रस्ट, जालंधर'' ने वर्तमान सेमीनार के आयोजन हेतु अपना सराहनीय सहयोग प्रदान किया। एतदर्थ आयोजन समिति आभार मानती है।

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THE GURUKULA KANGRI PHARMACY

R. K. Rawat*

The Gurukula Kangri Pharmacy is preparing ayurvedic medicines of best quality from last one century. Our products created their own place in the hearts of public due to best standard in quality and high ideality in preparation as mentioned in old Ayurvedic books. Now a days, we are one of the largest producer of ayurvedic drugs in India.

We are preparing almost 400 products. We are also developing new products related to various diseases. The potency of our medicines is far better and it is continuously increasing by modern techniques. The main branches of medicines, which we are preparing are Powders, Bhasm, Ras & Vati, Pak-Avaleh, Asava-Arishta, medicated oils and Ghritas.

Many new products are under consideration for preparation and testing. From the above mentioned products, the important products, which have leading sale are Bhimseni Surma, Chyavanprash, Payokil, Shilajeet, Madhumeh Nashini Gutika, Gurukul Chai & Gold preparations like Vrihat Vat Chintamani, Basant Kusumakar Ras & Yogendra Ras.

Since the unit is situated in the foothills of Himalaya, we are much capable to arrange fresh medicinal plants in comparison to other ayurvedic drugs producing units. The identification of purchased medicinal plants (crude drugs) is done by the expert ayurvedic doctors and chemists. After their approval, plants or medicinal material are used for preparation. This factor maintains the quality of products in one way.

The other way to improve quality is the methods of preparation of drugs. We are somewhat different with other ayurvedic drugs producing units because we are not compromising with the old methods of preparation of drugs given in standard books of Ayurveda. For this purpose, the strict supervision is done by the Aurvedic doctors.

The washing of crude material, crushing and grinding of materials or products is

*General Manager, Gurukula Kangri Pharmacy, Hardwar.

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done by the help of machines to make work faster and for other preparative methods, we are strict to traditional Ayurvedic standards.

Our products, specially Asavs and Arishtas are prepared in four to six months by self generating alcohol. So their potency against diseases has been recorded very high. Indeed, from all points, we proudly can say that due to high quality, our products are highly potent.

The packaging of medicines is done in different sizes of sealed packs. The packaging label has been printed by the weight of medicine, cost, year and batch number. Before purchasing medicine, it should be ensured that whether the seal is properly intact or not.

The drugs are sold through various agencies all over India.

The profit earned by the pharmacy is used for charity. The profit is donated for the education of students at Gurukula Kangri Vidyalaya (boys), Haridwar and Kanya Gurukula (girls), Dehradun.

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MATHEMATICS AND VEDIC LITERATURE

VIRENDRA ARORA

This paper intends to present a bird veiw of mathematical incidences in Vedic and post-Vedic literature; refer to Astrotree (see the inside cover).

It is a well established fact that Geometry of Aryans is predominated by arithmetic. Actually 'Hindus' algebraised geometry and the geometry of Indians have been developed out of necessity for the construction of Various *agnicitus, vedis, mandaps* etc. which are required for sacrificial ritual.

Present discussions are confined only to some of incidences of geometrical aspects related to Vedas and *Brāhamaṇas*.

The greatness of scientific discoveries made by ancient Indian seers is being progressively realized and admired by the modern scientific world. One of the scientific luminaries who has brought credit to this country for his outstanding discoveries and achievements in astronomy is the great Āryabhaṭa; generally known as the father of ancient Indian Mathematics. One of the great discoveries of this Indian astronomer of the 5th Century A.D. is that the stars are fixed and it is the earth which moves creating the phenomenon of the daily rising and setting of the Sun and the other planets and constellations by its motion on its axis as well as around the Sun.

भवंजरः स्थिरो भूत्वा वृत्यावृत्य
प्रतिदैवसिकौ उदयास्तमयौ
संपादपति ग्रह नक्षत्राणाम्।। (Arya Siddhanta)

According to social reformer Svāmī Dayānand Sarasvati : All science and technology that is or will ever be is contained in the seed form in the Vedas.

The term 'Veda' literally means 'Knowledge, All-Knowledge, 'Gnosis'. It is all reasoning and analysis. वेद विद (Veda Vid) means one who knows and acts up to the Vedas. It ultimately makes the man up righteous and such a man is ever beneficial to the people of the world at large.

Vedas are the words of God, revealed in the beginning of creation for the moral, spiritual and physical guidance and uplifts of humanity. They are replete with eternal truths and throw a light on various aspects of life to make a man perfect and ideal.

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God, out of his infinite source of knowledge reveals in the beginning of creation a part of it adequate for the requirements of the soul, its spiritual satisfactions, fulfilment of its thirst for truth and making its journey of life successful.

God is infinite, the soul is finite, therefore soul can not contain the infinite knowledge of God. God creates, sustains and dissolves the world.

Whenever the world is created God reveals the Vedas. This process is going on since eternity and will go on for ever.

There are 20380 verses in all the four Vedas namely Ṛg, Yajura, Sāma and Atharva as follows:

Ṛg Veda	10522
Yajura Veda	2006
Sām Veda	1875
Atharva Veda	5977

For correct interpretation of the Vedas, the words should be taken in their analytical sense, i.e., root-meaning should be taken into account of Vedic words. The Veda only the संहिता (the real text of the Vedas) as revealed by God in the beginning of Creation on *Agni, Vāyu, Āditya and Aṅgiras*, the four *Ṛshis* competent to receive God's Revelation.

Mathematics is the base of science. It is in the root of creation of the universe. There exists a motion in every particle of the universe which is related to mathematics; Times of sunrise, sunset, eclipses etc. are decided on the basis of calculations.

Also Mathematics has the highest place in Vedāṇg-astrology.

यथा शिखा मयूराणां नागानां मण यो यथा।
तद् वद् वेदागं शास्त्राणाां गणितं मुर्धनि स्थितम्।।
वेदागं ज्योतिष (याजुष)4

For accurate calculation of times of certain festivals and of times auspicious for acts of worship, good knowledge of arithmetic, plane, spherical trigonometry and geometry is required.

In Vedic religions, for purpose of worship, different types of altars are required. These altars have definite shape and area. They can be in the shape of square, triangle, semicircular according to worships for different purposes.

These acts of worships are found in the remotest towns. Several references are available in Ṛg Veda Saṃhita. The science of the contributions of altars take a mere specific

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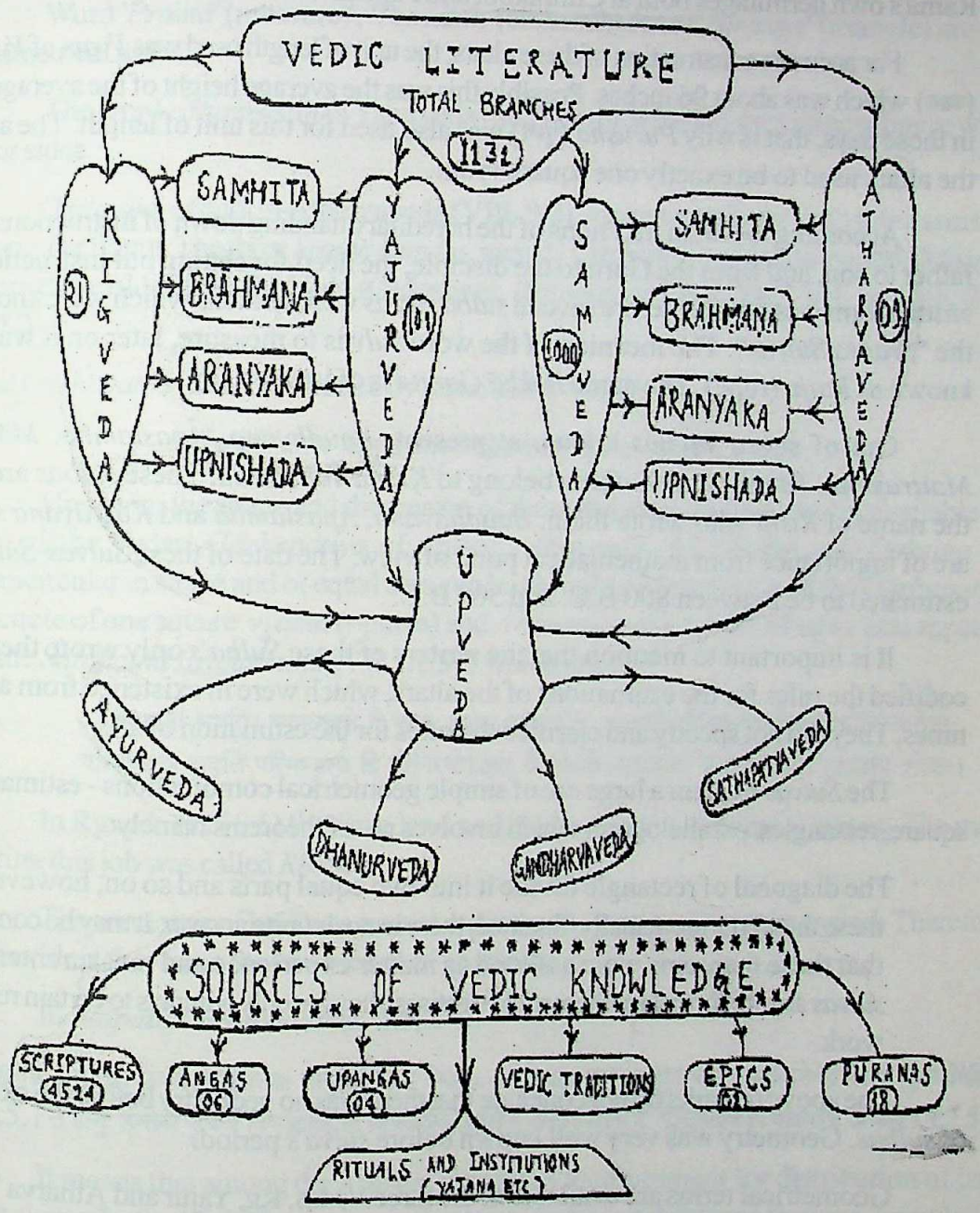

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form in the *Taitrireeya Saṃhita* and *Taitrireeya Brāhamaṇa*. It may be appropriate to mention that there was provision of these altars in the hermitage of sage Agasthya, and in Rama's own hermitages both at Chitrakoot and Panchavati.

For accurate construction of these altars, the unit of length used was *Vyam* of *Vyayam* (व्याम) which was about 96 inches. Possibly this was the average height of the average man in those days, that is why *Puruṣha* (पुरूष) was also used for this unit of length. The area of the altars used to be exactly one square *vyam*.

According to Hindu traditions of the hereditary handing down of instructions from father to son, and from the Guru to the disciple, the need for setting out instructions in written form was felt. Therefore several *śulba sūtras* were written, which were known as the "*Śrauta Sūtras*". The meaning of the word *sulv* is to measure, later on it was also known as *Rajju* (rope). The name *Rekhā Gaṇita* is of later origin.

Out of seven *śul'bas* known at present, *Baudhyāna*, *Āpastamba, Mānava, Maitrayaṇa, Vāraha* and *Vādhula* belong to *Ḳṛṣṇa Yajurveda*. These *śulbas* are after the name of *Rishi* who wrote them. *Baudhāyana*, *Apastamba* and *Kātyāyana śulvas* are of importance from mathematical point of view. The date of these *Śulvas-Sūtra* are estimated to be between 800 B.C. and 500 B.C.

It is important to mention that the writers of these *Śulba's* only wrote them and codified the rules for the estimations of the altars, which were in existence from ancient times. They did not specify and clarified the rules for the estimation of altars.

The *Śulvas* explain a large no. of simple geometrical combinations - estimation of square, rectangles, parallelogram which involves no. of theorems namely:

> The diagonal of rectangle divide it into two equal parts and so on, however how these theorems are actually obtained, there in no definite answer. It may be concluded that these theorems were asserted as matter experience and measurement. The *śulvas* are not formal mathematical treatises, they are only adjuncts to certain religious work.

The above remarks do not conclude that there was no geometry before the existence of *śulbas*. Geometry was very well known before *śulva's* period.

Geometrical terms are available in the three Vedas, Ṛg, Yajur and Atharva Veda.

The word *vṛttam* (RV I, 155.6) is used for circular wheel of the chariot. Also *trivrta* (RV I, 34, 9, 12) is used for the chariot of *As'vinau* which was triangular in shape.

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काऽऽसीत् प्रमा प्रतीमा किं निदानम् आज्यं किमासीत् परिधिः क आसीत्।
छन्दः किमासीत् प्रउगं किमुक्थं यद्देवा देवमयजन्त विश्वे।। Rg. X. 130.3.

Word '*Pramā*' (measure)', '*Paridhiḥ*' (circumference)', '*prauga*' (triangle) are all geometrical terms.

The word *catursraktiḥ* in Yajurveda (38.20) which means a rectilinear figure with four sides.

'*Tribhuja*' occurs in Atharvaveda (VIII, 9.2), meaning a figure with three arms in triangular figure. The three *Śrauta Kuṇḍas*, namely *Gārhapatya, Āhavanīya* and *Dakṣiṇa*, seem to be older than Ṛg Veda (Oldenberg, Religion ds Veda, SBE Vol. xxx No. 2 p. 348.)

The *Gārhapatya* fire is mentioned by name in Ṛgveda:

गार्हपत्येन सन्त्य ऋतुना यज्ञनीरसि। देवान् देवयते यज।। I.15.12.

However, the shape and dimension of these three scond fires are not available in any of the Vedas. *Gārhapatīya* of square, *Āhavanīya* of circular and *Dakṣiṇa* of semicircular in shape and of equal area can be thought of description of the *Gārhapatya* as circle of one square *vyama* (=pursa) and *Āhavanīya* of a square of same size appears in the *Śatapatha Brāhamaṇa* (VII.1.1.37) which states:

व्याममात्री भवति। व्याममात्रो वै पुरुषः पुरुषः प्रजापतिः प्रजापतिरग्निरात्मसंमितां तद्योनिं करोति परिमण्डला भवति परिमण्डला हि योनिरथोऽअयं वै लोको गार्हपत्यः परिमण्डल उ वाऽअयं लोकः।

In Ṛgveda (I 110.5) the farm land used to be divided into equal pieces. The man during this job was called *Rbhus*.

The surveying of land was a usual matter with Aryans of Ṛgvedic period. Therefore it is evident they have the knowledge of geometry.

In *Śatpatha Brāhmaṇa* (I, 2.5, 1-2)

देवाश्च वा असुराश्च। उभये प्राजापत्याः पस्पृधिरे ततो देवा ऽऽजुव्यमिवाऽऽसुरथ हासुरा मेनिरेऽस्माकमेवेदं खलु भुवनमिति।। I.2.5.1 ते होचुः हन्तेमां पृथिवी विभजामहै तां विभज्योपजीवामेति तामौक्ष्णैश्चर्मभिः पश्चात्प्राञ्चो विभजमाना अभीयुः। I.2.5.2.

It means that among the *Asuras*, the unit of measurement for distribution of land was skin of bullocks. It clearly shows that to measure and distribute the land, knowledge of geometry is imperative. *Śatpatha Brāhmaṇa* (III 5.1.26) gives the method of locating *cātvāla* which shows the progress in mensuration at the period.

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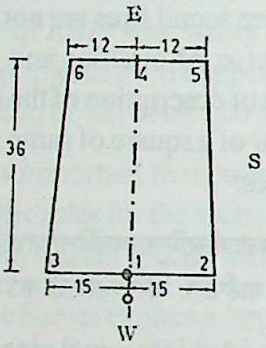
अथ श्म्यां च स्फ्यं चादत्ते। तद्य एष पूर्वार्ध्यः उत्तरार्ध्यः श्ङ्कुर्भवति तस्मात् प्रत्यङ् प्रक्रामति त्रीन् विक्रमांस्तचात्वालं परिलिखति सा चात्वालस्य मात्रा....।

It further confirms the state of knowledge from the detailed description given for the layout of the *Vedi* in *Śataptha Brāhmaṇa* (III.5.1.1-6)

तद्य एष पूर्वार्ध्यो वर्षिष्ठः स्थूणाराजो भवति। तस्मात् प्राङ् प्रक्रामति त्रीन् विकमांस्तच्छङ्कुं निहिन्त सोऽन्तःपातः। III.5.1.1.
तस्मान्मध्यमाच्छङ्कोः। दक्षिणा पञ्चदश विक्रमान् प्रक्रामति तच्छङ्कुं निहिन्त सा दक्षिणा श्रोणिः। III.5.1.2.
तस्मान्मध्यमाच्छङ्कोः। उदङ् पञ्चदश विक्रमान् प्रक्रामति तच्छङ्कुं निहिन्त सोत्तरा श्रोणिः। III.5.1.3.
तस्मान्मध्यमाच्छङ्कोः। प्राङ् षट्त्रिंशतं विक्रमान् प्रक्रामति तच्छङ्कुं निहिन्त स पूर्वार्धः। III.5.1.4.
तस्मान्मध्यमाच्छङ्कोः। दक्षिणा द्वादश विक्रमान प्रक्रामति तच्छङ्कुं निहिन्त दक्षिणोंऽसः। III.5.1.5.
तस्मान्मध्यमाच्छङ्कोः। उदङ् द्वादश विक्रमान प्रक्रामति तच्छङ्कुं निहिन्त उत्तरोंऽस एषा मात्रा वेदेः। III.5.1.6.

Also at the time of *Śatapatha Brāhamaṇa* the Pythagoras theorem was known as the parallelism of the sites 3,2 and 6,4 could not be achived without this.



(1) *Madhyama śaṅku* - central peg
(2) *Dakśiṇā śroṇi* - South-West corner
(3) *Uttarā śroṇi* - North-West corner
(4) *Pūrvārdhaḥ* - Estern central peg
(5) *Dakśiṇa aṁsa* - South-East corner
(6) *Uttarā aṁsa* - North-East corner

There are innumerable examples that confirm that even in pre-śulba period geometry was well known. Few of them are cited below:

1. The no. of bricks to construct different types of *Dhiṣṇyas* are mentioned in *Śatpatha Brāhamaṇs* for IX, 4.36-8.
2. The concept of construction a scrificial fire $7\frac{1}{2}$ square pursa area is described in *Śatapatha Brāhmaṇs* (X,2.3.5). It appears that geometrical knowledge was developed as per requirements of scarifical ritual, then it was converted in to *Sūtraform*.

The word *Akṣṇayā* which means hypotenuse or right angle triangle or diagonal of a square, rectangle or trapezium is known from the time of *Taitrĭya Saṁhita*. Also at the same time arithmetical calculations and geometrical constructions to find $\sqrt{2}$, $\sqrt{3}$ .....must have

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been developed.

There are no. of circumstantial evidences to show that the knowledge of geometry originated from the Vedas.

Details of different types of chariots and for different purposes is clearly mentioned in ṚgVeda (see Ṛg. V56,6, II,39,4). By the time of Yajur Veda, due to increased no. of different types of chariots, separate guild of chariot makers was developed. They were treated different from carpenter.

The proficiency in production of wheels confirms a good knowledge of geometry. The fixing of Spokes of odd or even nos. require knowledge of dividing the area of the circle into desired no. of small parts of equal areas (See Ṛg. I.164.13).

The *nābhi* of the wheel is located at the centre of the wheel showing thereby that construction for obtaining the centre of a given circle was known at that time.

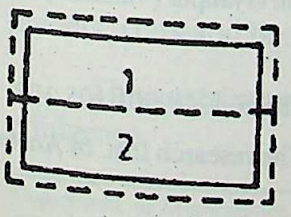
Ṛg Vedic Aryanas were also knowing the construction of drawing concentric circles of desired diameters, as *nābhi*, felly and the central hole must be concentric. Wells were covered by the owners (Ṛg. I, 55.8). Depending upon the shape of the well, knowledge of geometry is required to prepare the centre of month of the well.

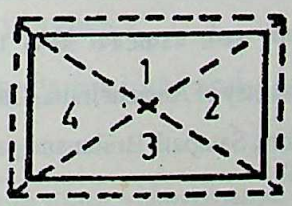
Construction of houses of different types are mentioned in Vedas, Nighaṇṭu (III.4) giving 22 names for a house in Vedas.

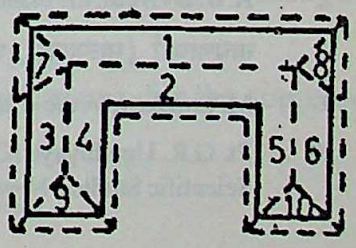
In Atharva Veda, it is clearly mentioned that a house is constructed according to measurements. A house is called *Mānasya Patnī* (Av.IX. 3.5;) also *yena chāsi mitā tvam* (Av.IX.3.9) *yas tvā sāle nimimāya* (Av.IX.3.11) etc.

The builders of the house is called *Māna-Pati*. The houses used to have many faces up to 10.

या द्विपक्षा चतुष्क्षा षट्पक्षा या निमीयते।
अष्टापक्षां दशपक्षां शालां मानस्य पत्नीमग्निर्गर्भ इवा शये।। (Av.IX.3.21)







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Description of sacrifice, ladies room, Kitchen etc. is clearly mentioned in Atharveda & Ṛgveda which confirms that geometry must have been developed at the Vedic time.

Town planning is also mentioned in Ṛgveda.

River-irrigation by constructing dams across them was known. The river is praised that it should flourish the country by filling the canals.

प्र पिन्वध्वमिषयन्तीः सुराघा आ वक्षणाः पृणध्वं यात शीभम्। (Rv.III.33.12)

Geometry seems to be necessary for the construction of even a very small dam across river.

In Atharva Veda, the town of gods called Ayodhya is described (AVX.2.31). This city should be circular in plan with eight rampart walls and nine doors.

अष्टाचक्रा नवद्वारा देवानां पूरयोध्या। (Av.X.2.31)

From the above discussion, it is evident that for production of chariots, constructions of houses, and their layouts in a town and also for accurate construction of different constituent parts for a variety of sacrifices, the Aryans of the *Vedic* and *Brāhmaṇa* period might have developed the geometrical knowledge.

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BHĀSKARĀCĀRYA : THE GREATEST SCHOLAR OF VEDIC PRIME KNOWLEDGE

S. L. SINGH AND RAMESH CHAND*

In Ancient India, there had been 18 authorities (immortal beings or sages) of Vedic supreme knowledge :

सूर्यः पितामहो व्यासो वसिष्ठोऽत्रिः पराशरः।
कश्यपो नारदो गर्गो मरीचिर्मनुरङ्गिराः।।
लोमशः पुलिशश्चैव च्यवनो यवनो भृगुः।
शौनकोऽष्टादशैवैते ज्योतिःशास्त्रप्रवर्त्तकाः।।

1. *Sūrya*, 2. *Pitāmaha* 3. *Vyāsa*, 4. *Vasiṣṭha* 5. *Atri*, 6. *Parās'ara*, 7. *Kas'yapa* 8. *Nārada*, 9. *Garga*, 10. *Marici*, 11. *Manu*, 12. *Aṅgirā*, 13. *Lomas'a or Romas'a*, 14. *Pulis'a or Paulasiya*, 15. *Cyavana*, 16. *Yavana*, 17. *Bhṛgu*, 18. *S'aunaka*.

These various names stand perhaps for different schools of astronomy.

In ancient India, the word astronomy was used for the computation of celestial luminaries. What special place the astronomical culture had at that time could be visualised from the following verse of *Vedaṅga Jyotiṣa* :

यथा शिखा मयूराणां नागानां मणयो यथा।
तद्वद्वेदांगशास्त्राणां ज्योतिषं मूर्धनि स्थितम्।।

(As the crests of peacocks and the jewels of serpents, astronomy stays at the head of all the *Vedaṅga Śāstras*).

Astronomy developed in ancient India as an adjunct to the rules of worship laid down in the Vedic art of living. So it came to be classified as one of the six *Vedāṅgas* or limbs of the Vedas and was known by the name of *Jyotiṣa*, which literally means the science of celestial bodies. In a secondary and later sense it stood for the study of planets, stars and heavenly bodies.

Thus astronomy stands for light and, in the highest sense, for the supreme effulgence.

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Bhāskarācārya composed a great astronomical treatise *Śiddhānta Siromaṇi* (SS) in 1150 A.D. at the age of 36. The SS gives a complete account of Indian astronomy up to 1200 A.D. This holy *Siddhānta* reflects the Vedic supreme knowledge of *Ācārya Bhāskara*. In this *Grahaganita* he says :

वेदास्तावद् यज्ञकर्म्मप्रवृत्ता यज्ञाः प्रोक्तास्ते तु कालाश्रयेण।
शास्त्रादस्मात् कालबोधो यतः स्याद् वेदाङ्गत्वं ज्यौतिषस्योक्तमस्मात्।।
शब्दशास्त्रं मुखं ज्यौतिषं चक्षुषी श्रोत्रमुक्तं निरुक्तं च कल्पः करौ।
या तु शिक्षाऽस्य वेदस्य सा नासिका पादपद्मद्वयं छन्द आद्यैर्बुधैः।।
वेदचक्षुः किलेदं स्मृतं ज्यौतिषं मुख्यता चाङ्गमध्येऽस्य तनोच्यते।
संयुतोऽपीतरैः कर्णनासादिभिश्चत्तुपाऽङ्गेन हीनो न किञ्चित्करः।।

This means:

Yajña is fundamental to vedic culture, and the time of a particular *Yajña* is of vital importance. So the astronomy occupies the top place amongst the limbs of Vedas. If one compares the Vedas with a human body then *Śabda Śāstra* (the science of grammar) is the face, astronomy stands for eyes, the *Nirukta* for ears, the *Kalpa* for hands, the Sikṣa for the nose and the *Chanda Śāstra* for legs. Since a body without eyes is hollow, the astronomy remains the top most limb of the *Vedapuruṣa*.

LIFE HISTORY OF *BHĀSKARĀCĀRYA*

The great astronomer, astrologer and top-notch mathematician saw the light in Śaka 1036 (1114 A.D.) at the village Bid (Presently known Bijapur in Karnataka). The village is located near mountain Sahya, famous for the central place of divine scholars in those days. He was born in *Śaṇḍilya Gotra* (lineage). His father *Maheśvara* was a great scholar and priest. His wisdom was hereditary as he belonged to a wise race and scholarly family. He himself has written in *Golādhyāya* of SS :

रसगुण पूर्णमही समशकनृपसमयेऽभवन्ममोत्पत्तिः।
रसगुणवर्षेण मया सिद्धान्तशिरोमणी रचितः।।
आसीत् सह्यकुलाश्रितपुरे त्रैविद्यविद्वज्जने।
नानासज्जनधाम्नि विज्जडविडे शाण्डिल्यगोत्रो द्विजः।।
श्रौतस्मार्तविचारसारचतुरो निःशेषविद्यानिधिः।

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साधूनामवधिर्महेश्वरकृती दैवज्ञचूड़ामणि:।।
तज्ज्स्तच्चरणारविन्दयुगलप्राप्तप्रसाद: सुधी:।
मुग्धोद्बोधकरं विदग्धगणकप्रीतिप्रदं प्रस्फुटम्।।
एतद्व्यक्तसदुक्तियुक्तिबहुलं हेलावगम्यं विदाम्।
सिद्धान्तग्रथनं कुबुद्धिमथनं चक्रे कविभास्कर:।।

The following inscription (of about 1210 A.D.) due to *Caṅgadeva*, the grandson of *Bhāskarācārya* was first discovered by Dr. Bhau Daji at the village Patana near Devgiri (Sivanatha Jhārakhandi, *Bhārtīya Jyotiṣa*) :

शाण्डिल्यवंशे कविचक्रवर्ती त्रिविक्रमोऽभूत्तनयोऽस्य जात:।
यो भोजराजेन कृताभिधानो विद्यापतिर्भास्करभट्टनामा।।
तस्मात् गोविन्दसर्वज्ञो जातो गोविन्दसन्निभ:।
प्रभाकर: सुतस्तस्मात् प्रभाकर इवापर:।।
तस्मान्मनोरथे जात: सतां पूर्णमनोरथ:।
श्रीमन्महेश्वराचार्यस्ततोऽजनि कवीश्वर:।।
तत्सूनु: कविवृन्दवन्दितपद: सद्वेदविद्यालता
कन्द: कंसरिपुप्रसादितपद: सर्वज्ञविद्यासंद:।
यच्छिष्यै: सहकोऽपि नोविवदितुं दक्षो विवादी क्वचित्
श्रीमान्भास्करकोविद: समभवत् सत्कीर्तिपुण्यान्वित:।।
लक्ष्मीधराख्योऽखिलसूरिमुख्यो वेदार्थवित्तार्किकचक्रवर्ती।
क्रतुक्रियाकाण्डविचारसारविशारदो भास्करनन्दनोऽभूत।।
सर्वशास्त्रार्थदक्षोऽयमित्रि मत्वा: पुरादत्त:।
जैत्रपालेन यो भीत: कृतश्च विवुधाग्रणी।।
तस्मात् सुत: सिंघणाचक्रवर्विदैवयज्ञवर्योऽजनि चंगदेव:।
श्री भास्कराचार्यनिबद्धशास्त्रविस्तारहेतो: कुरुते मठं य:।।
भास्कर रचित ग्रन्था: सिद्धान्तशिरोमणि प्रमुखा:।
तद्वंश्य कृताश्चान्ये व्याख्येया मन्मठे नियमात्।।

The following is the genealogy of *Bhāskarācārya* :

Trivikarama → *Bhāskarabhaṭa* → *Govinda* → *Prabhākara* → *Manoratha* → *Maheśvara* → *Bhāskarācārya* → *Lakṣhmidhara* → *Caṅgadeva*.

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EDUCATION

Bhāskarācārya received education at the feet ot his learned father since his childhood. He studied *Vedas, Purāṇas, Upaniṣadas, Rāmāyaṇa, Mahābhārata, Gītā, Siddhāntas* and several other works of *Varāhamihira, Brahmagupta, Śridhara, Lalla, Padmanabha, Pṛthudakasvāmi, Muñjāla* and *Śrīpati* and others. He offered the humble criticism in the works of his forescholars, whenever he found errors. For, example, he criticised *Lalla* in the following stanza of *Golādhyāya :*

दुष्टं कन्दुकपृष्ठजालवदिलागोले फलं जल्पितं
लल्लेनास्य शतांशकोऽपि न भवेद्यस्मात्फलं वास्तवम्।
तत्प्रत्यक्षविरुद्धमुद्धतमिदं नैवास्तु वा वस्तु वा
हे प्रौढा गणका विचारयत तन्मध्यस्थबुद्ध्या भृशम्।।

(Pt. Kedardatta Joshi (Ed.) Golādhyāya of Bhāskarācārya, Motilal Banarsidas, Delhi, 1988).

The Following stanza shows the deptth of his knowledge :

अष्टो व्याकरणानि षट् च भिषजां व्याचष्ट ताः संहिताः
षट् तर्कान् गणितानि पञ्च चतुरो वेदानधीते स्म यः।
रत्नानां त्रितयं द्वयं च बुबुधे मीमांसयोरन्तरं
सद्ब्रह्मैकमगाधबोधमहिमा सोऽस्याः कविर्भास्करः।।

This means: It was the poet *Bhāskara,* who had read all the eight volumes of grammar and the six branches of medical science with their commentaries, who had studied the six logics, five branches of mathematics and the four Vedas, a trial of three ratnas, who could distinguish between the two *Mimānsās* (branches of philosophy), only the great *Brahmā* (creater of the universe) could know the depth of his knowledage. [Thus he had not mustered only Mathematics but was well versed in many branches of knowledge.] Also:

भास्करीय गिरां सारं भास्करो वा सरस्वती।
चतुर्मुखोऽधवावेति विदुर्नान्ये तु माद्शाः।।

This means: The essence of the words of *Bhāskara* could be understood either by *Bhāskara* himself or by *Sarasvatī* , the goddess of learning or by *Brahmā* , the creater of the universe, and not by (ordinary persons like us).

STYLE

In the beginning of his *Grahaganita* he says :

कृती जयति जिष्णाजो गणकचक्रचूड़ामणि—
र्जयन्ति ललितोक्तयः प्रथिततन्त्रसद्युक्तयः।
वराहमिहिरादयः समवलोक्य येषां कृतीः
कृती भवति माद्शोऽप्यतनुतन्त्रबन्धोऽल्पधीः।।

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This shows that *Bhāskarācārya* was influenced with the poetic style of *Brahmagupta* and *Ācārya* Varāhamihira . He followed their poetic style in his writings, and the climax of his poetic beauty is attained in his *Līlāvatī* WORKS *Bhāskarācārya* was inspired by the holy sermon of *Gītā* :

कर्मण्येवाधिकारस्ते मा फलेषु कदाचन्।

(Vināyaka Gaṇesa Āpate, Bījaganita of Bhāskarācārya Ānanda Āsarama Press, Poona, 1930). He composed *Siddhānta Śiromaṇi* without any imperial support. His commentators also follwed him. His SS is a comprehensive work in four parts.

1. *LĪLĀVATĪ* (arithmetic, algebra, gemoetry, mensuration, and a bit of trigonometry): It contains about 280 verses including half verses.

2. *BĪJAGAṆITA* (algebra) : It contains 187 verses.

3. *GOLĀDHYĀYA* (theory of sphere or trigonometry inculding spherical trigonometry.) It deals with the astronomical basis of mathematical calculations about grahas (planets). It contains 563 verses.

4. *GRAHAGAṆITA* (planetary motion) : It contains 447 verses.

HIS OTHER WORKS

1. *KARAṆA KUTŪHALA :* It is an astronomical work containing 138 verses. He composed this book in *śaka* 1105 (1183 A.D.) when he completed the age of 69 years.

2. SARVATOBHADRA YANTRA

3. VASITHTULYA

BHĀSKARĀCĀRYA : THE GREAT ASTRONOMER

Spottiswoode (S.R. Sinha, Mathematics Educalion(Siwan), 15 (1981), p.82) remarks :

It must be admitted that the penetration shown by *Bhāskara* in his analysis is in the highest degree remarkable, that the formulae which he establishes and the methods of establishing them bear more than a resemblance ..... they bear an analogy ..... to the corresponding process in modern astronomy, and the majority of scientific persons will learn with surprise the existence of such a method in the writings of so distant a period and so distant a region.

Golādhyāya and *Grhagaṇita* of *Bhāskarācārya* give a complete account of Indian astronomy nearly up to 1200 A.D. A thorough discussion of the obliquity of the ecliptic and its effect on the equation of time reflects a great astronomical maturity of the *Ācārya.*

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For the measurement of time *Bhāskara* gives an account of time scale *yuga* system as described and used by ancient Indian seers and astronomers. He gives the following stanza for the computation of planetary positions of the planets, when the present *Kaliyuga* begins.

खाद्रिरामाग्नयः ३३७० क्कग्निरामाङ्कका ९३३१
वेदवेदाङ्कचन्द्रा १९४४ विलिप्ताः क्रमात्।
षड्साङ्गब्धयो ४६६६ ऽङ्गभ्रवेदाब्धयो ४४०६
वेदषट्काभ्रभूपाभ्रभूसंमिताः १०१६०६४।।
वेदचन्द्रद्विवेदाब्धिनागाः ८४४२१४ कर–
द्व्यब्धिवेदाब्धिशैला ७४४४२२ भवेयुः कुजात्।
द्वापरान्तधृवाश्चक्रशुद्धास्तथा
सूर्यतुङ्गेन्दुतुङ्गेन्दुपातोद्भवाः।।

This means :

Mars	Mercury	Jupiter	Venus	Saturn	Solar Apogee	Lunar Apogee	Ascending lunar node	
11R	11R	11R	11R	11R	2 R	4 R	5R	Rāśis
29°	27°	29°	28°	28°	17°	5°	3°	Degrees
3'	24'	27'	42'	46'	45'	29'	12'	Minutes
50"	29"	36"	14"	34"	36"	46"	58"	Seconds

For the computation of *ahargaṇa cahargaṇa* literally means aggregate of civil days) since the beginning of creation up to the end of *Śaka* king he gives the following verse in his *Grahaganita:*

याता षण्मनवो युगानि भमितान्यन्यद्युगाङ्घ्रित्रयं
नन्दाद्रीन्दुगुणा (३१७९) स्तथा नृपशकस्यान्ते कलेर्वत्सराः।
गोद्रीन्द्रद्रिकृताङ्कदस्रनगगोचन्द्राः (१९७२९४७१७९)
शकाब्दान्विताः सर्वे सङ्कलिताः पितामहदिने स्युर्वर्तमाने गताः।

Secondly, he gives the following verse for the computation of *ahargaṇa.*

कथितकल्पगतोऽर्कसमागणो
रविगुणो गतमाससमन्वितः।
खदहनै ३० र्गुणितस्तिथिसंयुतः

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पृथगतोऽधिकमास १५९३३००००० समाहतात्।।
रविदिना १५५५२०००००००० सगताऽधिकमासकै:
कृतदिनै: सहितो द्युगुणो विधो:।
पृथगत: पठितावम २५०८२५५०००० संगुणाद्
विधुदिना १६०२९९९०००००० सगतावमवर्जित:।।
भवति भास्करवासरपूर्व्वको दिनगणो रविमध्यमसावन:।
अधिकमासदिनक्षपशेषतो द्युघटिकादिकभत्र न गृह्यते।।

In his *siddhānta Śiromaṇi, Bhāskarācārya* has given several techniques for the computation of planetary positions at any moment. He gives a wonderful verse for the computation of planetary position of all the planets :

दिग्भि: १०१४६१ नगाष्टनगभूतिथिभि: क्रमेणा १५१,७८७।
देवाष्टखाङ्कशशिभि: १९०८३३ च रसाग्निवेद–
सिद्धै: २४४३६ खखाब्धिदहनाभ्रयमेन्दुभिश्च १२०३४००।।
भूपाब्धिलोचनरसै: ६२४१६ खखखाभ्रनन्द–
नन्दाश्विभि: २९९०००० गगनखाभ्रगजाङ्कनागै: ८९८०००।
खाभ्राष्ट पङ्कजधृतिप्रमितै: १८८६८०० च भक्ताद्
भागादिकानि हि फलानि रवे: राकाशात्।।
विघो: फलं खाश्विगुणं विधेयं ग्रहध्रुवा: स्वस्वफलै: समेता:।
ते वा भवन्ति द्युचरा: क्रमेण भागादिक: स्यात् फलमेव भानु:।।

For the computation of solar and lunar eclipses he has given scientific methods.

BHĀSKARĀCĀRYA : THE GREATEST MATHEMATICIAN

Bhāskarācārya was one of the greatest mathematician who appears to have established first the modern theory in the convention of signs (minus x minus = plus) in the following verse:

स्वयोरस्वयो: स्वं वध: स्वर्णघाते क्षयो भागहारेऽपि चैवं निरुक्तम्।

(A study of of *Bhāskarācārya* , Chaukhamba Orientaliya, Varanasi, 1979, p.51).

Bhāskarācārya , in *Līlāvatī*, gave formulae for the surface area and volume of sphere and volume of frustum of a pyramid. His logistics include addition, subtraction, multiplication, division, squaring, cubing, square-root, cube- root of numbers. It also deals with the algebra of zero, the rule of three (which was later called Golden rule in the West) permutations, combinations and *Kuṭṭaka.* The section on geometry deals with triangles, quadrilaterals, circle, cyclic sphere, trigonometricals functions etc.

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In *Līlāvatī Bhāskarācārya* picked up the nature as the store house of wisdom. The following stanza reflects *Bhāskara* s love to nature while posing a question on quadratic equation :

यातं हंसकुलस्य मूलदशकं मेघागमे मानसं
प्रोड्डीय स्थलपद्मिनीवनमगादष्टांशकोऽम्भस्तटात्
बाले बालमृणालशलिनि जले केलीक्रियालालसं
द्दष्टं हंसयुगत्रयं च सकलां यूथस्य सङ्ख्यां वद।।

That is :

O! tender girl, on the advent of rains out of the swans in a lake, ten times the square-root of their number went to *Manasarovar, 1/8th* (of them) moved to *Sthala Padmini* (a forest). The remaining three pairs were engaged in water sports. Tell the total number of swans.

We fully share what Datta and Singh say: It is certain, however, that after the 12th century very little origional work was done in India, commentaries on older works were written and some new works brought out, but none of these had sufficient merit as regards exposition or subject matter, so as to refine and extend the works of *Bhāskara* II, which had held undisputed sway for almost nine centuries (as standard text books).

Bhāskarācārya 's *Bījagaṇita* is akin to our present day secondary school algebra and a little more. The *Bījagaṇita* gives general methods for solving linear quadratic indeterminate equations which were given after 500 years by western scholars-such as Fermat (1601-1665), Lagrange (1736-1813) and Euler (1707-1783). *Bhāskara* gave the *cakravāla* method for obtaining the general integral solution of the indeterminate equation $Nx^2 + 12 = y^2$. This is considered a big achievement in Number Theory during the golden period.

Bhāskarācārya gave the concept of mathematical infinity (*Khahara*) and the analysis concerning the limit of an algebraic expression is implicited in his examples in *Līlāvatī* and *Bījagaṇita.* In the history of mathematical sciences, this is perhaps the greatest contributions to the human civilization.

Martin Ohm (1628) says : 'If a is not zero but b is zero, then the quotient $\frac{a}{b}$ has no meaning for 'the quotient multiplied by zero gives only zero and not a, as long as a is not zero" (Datta and Singh, History of Hindu Mathematics, p. 246). That is even 19th century modern mathematicians were not clear in such concepts.

Indeed, it appears that *Bhāskarācārya* was the first to introduce the concept

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of limit. He has used this technique in his astronomy.

The recent investigations by S.R. Sinha. [The Mathematics Education (Siwan) 15 (1981), 69-82] and C.N. Srinivasienger [History of Ancient Indian Mathematics, World Press Private Ltd., Calcutta 1967] have shown that *Bhāskarācārya* developed the concept of Differential Calculus but certainly not in the modern language. The above notion, 500 years before Newton and Leibniz appears in his famous treatise SS. Indeed, for the exact computation of daily motion of planets, *Bhāskara* introduced the concept of *Tatkālikagati* (instantaneous velocity) by dividing the day into a large number of small intervals and compared the positions of the planets at the end of successive intervals.

If y and y' are the mean anomalies of the planets at the end of concecutive intervals, then according to *Bhāskara*:

बिबार्धस्य कोटि ज्यागुण स्त्रिज्याहर: फलं दोर्ज्यायोरंतरं।

(The product of the cosine of the semidiameter by the element of the radius gives the difference of two sines). This result is equivalent to :

$$H \sin y' - H \sin y = (y' - y)\, H \cos y.$$

In modern language, this is equivalent to

$$\delta (\sin y) = \cos y \; \delta y$$

In relation to the basic idea of ROLLE'S THEOREM *Bhāskarācārya* has gone deeper into Differential Calculus in SS and suggests that the differential coefficient vanishes at an extreme value of the function. In *Grahaganita (spastādhikāra) he says :*

यत्रग्रहस्य परमंफलं तत्रैबगतिफलाभावेनभवितव्यं

.....यतोवक्रारंभे वक्रत्यागेच गति: पूर्णभवति

(where the planets motion is an extremum, there the fruit of motion is absent, i.e., the motion is stationary. At the commencement and end of retrograde motion, the apparent motion of the planets vanishes). That is, the derivative vanishes at a point of maxima or minima.

BHĀSKARĀCĀRYA : THE GREATEST SCIENTIST

Regarding the gravitational power of the earth *Bhāskarācārya* criticised the *Buddhist* philosophy (Pt. Kedardatta Joshi (Ed). *Goladhyaya* of *Bhāskarācārya*. Motilal Banarsidas, Delhi, 1988) and gave the appropriate theory in the following stanza of his *Golādhyāya*:

"आकृष्टिशक्तिश्च महीतया यत् खस्थं गुरुंस्वाभिमुखं स्वशक्त्या।
आकृष्यते तत्पततीव भाति समे समन्तात् क्व पतत्वियं खे।। "

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("पृथ्वी में गुरूत्वाकर्षण शक्ति है जो आकाश के गुरू पदार्थो को खींचकर अपने धरातल में ले आती है। पृथ्वी में यह एक ईश्वरदत्त गुरूत्वाकर्षण शक्ति सदा वर्तमान रहती है, जो आकाश में अपनी सीमा के भीतर भी किसी भी गुरू पदार्थ को खींचकर अपने धरातल में ले आती है और साधरण द्रष्टा को आकाशीय पदार्थों का पृथ्वी में गिरने का सा अनुभव होता है"।)

THE POET

The SS reflects that Ācarya was keenly interested in poetry. He describes *Ṛtuvarnaṇam* and *Śrigonnativāsanādhyāya* in *Golādhyāya* in the form of poetry which is embalished with figures of speech such as simile, metaphor, and hyperbole etc. In *Līlāvātī*, he has given several examples from the nature, which show that he understood nature as the store house of wisdom. Indeed he was a great lover of nature.

DEATH

Bhāskarācārya (Trans.), the great mathematician, astronomer and astrologer left for his heavelny abode in 1193 A.D. (K.S. Patavardhan, S.A. Naimpally and S. L. Singh, *Līlāvatī* of *Bhāskarācārya* , Motilal Banarsidas, Delhi 2001). Even we Indian foregot to celebrate the 800th anniversary of the *'SIDDHĀNTA ŚIROMAṆI'* which is one of the greatest astronomical works on this earth.

We assest without any hesitation that *Bhāskarācārya* was a great mathematician, astronomer, scientist, philosopher, poet and a great scholar of Vedic prime knowledge and he made significant contributions to lay the foundation stone of modern mathematics, modern astronomy, arithmetic, algebra, trigonometry, geometry, calculus and the concept of limit. He will remain the source of knowledge and inspirations for mathematicians and the generations to follow.

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Book Review
by *V. Madhukar Mallayya* (Trivandrum)

LĪLĀVATĪ OF BHĀSKĀRĀCARYA

A Treatise of Mathematics of Vedic Tradition

Translated by:
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Bhāskara II or Bhāskārācārya was one of the most illustrious astronomer-cum-mathematician of the 12th Century A.D. The Ācārya has Composed standard treatises on astronomy, mathematics and astrology, namely the Siddhānta Śiromaṇi-Līlāvatī-Bījagaṇita, Vāsanābhāsya, Karaṇakutūhala, Bhāskara Vyavahāra and Vivahapaṭala. Of these, the Siddhānta Śiromaṇi-Līlāvatī- Bījagaṇita combine from the magnum opus of the Acārya. Because of the high merit of the contents, the works of the Acārya became the main source of reference for mathematicians and astronomers of later period. Most of the other works composed during that period were over shadowed and driven to obscurity by the remarkable brilliance of Bhāskāra's analytic exposition.

The Līlāvatī is a basic treatise on Mathematics encompassing the pre-requisites for a wider study of astronomical science and various topics useful for domestic computations having applications in day to day life. This delightful treatise was composed by Bhāskara at the age of 36 in the year 1150 A.D. Līlāvatī in Sanskrit means 'charming' or 'beautiful'. As such the title of this charming work of mathematical enunciations and beautiful illustrations is quite appropriate. Faizi's translation of the work into Persian narrates a touching story relating to a misfortune that be fell on the beautiful daughter Līlāvatī of Bhāskarācārya on account of which she had to remain unmarried for ever in her life. In order to console his unfortunate

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daughter and keep her dejected mind fully occupied in learning mathematics, the intelligent father started teaching her mathematics from the basic level in an interesting manner and composed the quintessence of his deliberations in the form of beautiful verses and named it Līlāvatī. On completion of this task, the Ācārya seems to have totally succeeded in bringing back joy and happiness to his daughter and probably because of this experience Bhāskara generalizes and happily remarks in the concluding stanza that anyone who masters the Līlāvatī will become happy and prosperous. This message is embodied in the verse-

येषां सुजातिगुणवर्गविभूषितांगी
शुद्धाखिलव्यवहृतिः खलु कंठसक्ता।
लीलावतीह सरसोक्तिमुदाहरन्ती
तेषां सदैव सुखसंपदुपैति वृद्धिम्।।

"Joy and happiness is indeed ever increasing in this world who have Līlāvatī clasped to their throats, decorated as numbers are with neat reduction of fractions, multiplications and involutions, pure and perfect as they are the solutions and tasteful as in speech which is exemplified.

The book under review is an enlargement of the Phadke's Marathi version of the Līlāvatī, translated into English and expounded in modern mathematical language. The work is divided into thirty four chapters followed by the index of verses and a subject index.

Bhāskara's first invocation stanza to lord Ganesa is prefixed to Chapter 1. After invoking the blessings of the Lord, Bhāskara proceeds to describe the units of value (currency denominations) that was in vogue in the country during that period, followed by tables of commonly used weighing and measuring units. An understanding of size and weights of objects and measures of space around is a basic need of man and absolutely essential in the science of computation. Knowing fully the all pervasive nature of measures, the section on weights and other measures has been separately dealt with and prefixed to the actual treatise by the Ācārya. The authors of the book under review have given in Chapter 1, their detailed comments on each of the measures listed down by Bhāskara. The actual mathematical treatise commences with Bhāskara's second invocation stanza, followed by the numeration terminology for dealing with numbers running upto 18 places of figures. The

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contents of the book from Chapter 2 to Chapter 34 may be broadly classified into three sections.

i) Aṅkaganita dealing with real numbers and various operations and determinations involving numbers.

ii) Kṣetraganita dealing with the geometrical figures both plain as well as solid along with some of the applications in domestic computations, and

iii) Kuṭṭaka dealing with indeterminate equations of first degree and methods of finding their solutions. Without any such classification, the various topics coming under the preview of these three are divided into thirty three chapters and discussed briefly in the modern language by the authors. These Chapters are entitled, Place Value of Digits; Addition and subtraction; Method of multiplication; Division; method of finding squares; Square root; Method to find the cube; Cube roots; Eight operations on fractions; Addition and subtraction of fractions; Multiplication of fractions; Division of fractions; Squares, Cubes, Square roots and Cube roots of fractions; Eight rules concerning zero; Reverse process; To find an unknown quantity; Method of transition; Square transition; Quadratic equations; The rule of three; Inverse proportion; The rule of five; Rule for barter; Simple interest; Combinations; Progression (Series); Mensuration; Volume; Wood cutting; Volume of a heap of grain; Shadows; Pulverization; Concatenation (Permutation, Partitions etc.). The authors have added their own comments after giving translation of various verses in simple form in lucid English, the style of which is easy to follow and any student interested in this field can get a clear idea in the modern terms of the content of the verses. However, in the case of some of the verses the authors have deliberately done away with literal translations and the renderings of such stanzae give only mathematical formulae by expressing the quintessence of the content of Bhāskara's enunciations in an intelligible form using modern mathematical symbolic notations. The authors have succeeded in their efforts to bring out clearly and accurately the mathematical knowledge embedded in the words of Bhāskara and they have expounded the contents in modern language in an appreciable form which will definitely enable the modern reader to study the sūtras without any strain and without loss of Bhāskara's efforts to make the learning of mathematics interesting and charming. The translators efforts to make "faithfulness" and "beautifulness" to go together, by narrowing down the gap between Sanskrit verses and their English

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renderings without drifting far away from the literal beauty of the verses and at the same time remaining deeply faithful to the mathematical message latent in them, is quite appreciable. For instance, Bhāskara's analysis of indeterminate equations is fully explained in modern language using continued fracions. To mention another noteworthy message, the Acārya's rules concerning division by zero is explained using modern concept of limits thereby making one hear the knocking at the doors of calculus and infinitesimals by the Ācarya as early as 12th century AD. However historical remarks and notes are not many and if such comments are also included in the next edition of the book, it will be widely appreciable. The natural order of words in a verse (anvaya) may also be given prior to their English translation so that the readers having some knowledge of Sanskrit can enhance their enjoyment of learning basic mathematics by riding smoothly through the medium of poetry and thus appreciate fully the beautiful poetic flights of a scientific literature.

The Pubalishers of the book also deserve congratulations for their nice job in bringing out the book in neat printing, free from errors, with overall beauty in get up and lay out. This modern rendering of the Līlāvatī with brevity in words of mathematical wisdom maintaining throughout the clarity in conveying the mathematical message of Bhāskara continues to give pleasure and insight to every class of readers from children to the aged who are interested in both Sanskrit and Mathematics.

So, dear reader, do read this charming work, own a copy and adorn your personal library which will be a valuable addition to your own personal collections in the sense that, some day, some child, some where who happens to read the delightful Līlavātī of Bhāskarācārya, may get inspired and motivated by the charm of the words of Bhāskarā II and hopefully a new Bhāskarācārya (Bhāskara III) may emerge in this Land of Wisdom which has bequeathed to us a bright galaxy of eminent astronomers and mathematicians.

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Abstracts
of
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Department of Mathematics and Statistics

GURUKULA KANGRI VISHWAVIDYALAYA
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1
YAMATAIZATION

B. S. Yadav

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Every language has a grammar of its own. Out of all languages, it is well known that Sanskrit has the most elaborate and precise grammar. Even its poetical composition is metrical. Sanskrit verse is regulated by quantity and not by accent. Though Agni Purana deals with the subject exhaustively, the Pingalachhandas-sastra by Pingalacharya has been in popular use. In this case, chhands are meterd by eight Ganas or syllabic feet. There is a *sutra* (formula): yamatarajabhansalagam which serves as a handy tool to memorize these eight ganas. The talk concentrates on the history and composition of this formula with various ramifications in mathematics, particularly with its connection with Memory Wheels and other solved and unsolved problems. Yamataization is a name coined to present the subject in a more general framework.

2
CALENDER SYSTEM IN ANCIENT INDIA

G. S. Pandey

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Ever since the dawn of Vediec civilization on this planet the counting of time has considered as the highest need of the society. During the Rigvedic period a solar year was recognized as formed of twelve months and each month of 30 days. Vedang Jyotisa (1400 B.C.) introduce the concept of Yuga system of five samvatsars (years) and made a clear distinction between a tithi (lunar) day, Savan and Solar day. Also, in order to account for a solar year, the system of intercalary month was introduced.

We present a short account of the main fifteen-calendar system introduced in India from time to time. In particular, we trace the history of Vikrma Samvat (ignited from 57 B.C.) formulated on the basis of Surya Siddhanta, which is the only calendar system being used, with some variations, almost everywhere in India, while the Government of India has declared an entirely fabricated solar form of Saka Samvat as the national Calendar.

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3
BAHAUDDINAL–CAMILI 'S ESSENCE OF MATHEMATICS- AS A TEXT BOOK IN MADRASAS

S. M. R. Ansari

Roshan Villa, Muzammil Manzil Compound, Dodhpur, Aligarh

Al–^{c}Amili was an Iranian mathematician of 16th century. His Arabic treatise : *Khulas*atul H'isab* was very popular in all Iranian and Indian Madrasas. It was translated into Persian in Medieval India, and also translated into German and French in the first quarter of 20th century. The Russian translation is also now available. Many commentaries were written in Persian by Medieval Indian scholars. This is just a brief introduction. The significance of the work is also due to the fact, that al–^{c}Amili was the first mathematician who stated in his book the so-called Fermat theorem: $x^n + y^n = z^n$? for n>2.

4
MAHAVIRA'S METHOD OF FALSE POSITION

Kanahiya Jha

Dep. of Mathematics, L.B.S.M College, Jamshedpur

History is essential for us, so that we can assess, how much we are indebted to our fore fathers. The great master has given us different ideas. If we develop these ideas to suit our needs it will be a real salutation to them. For this we have to translate these ideas in modern terms, so that it can be fit for use in solving modern problem. We will discuss briefly some ideas and rules of Mahavir (9th Century) from his work ganit sara sabgrah, about rule of false position, method of obtaining unit fraction for any given fraction some interesting aspects of series etc.

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5
MATHEMATICS IN VEDIC TRADITION

Virendra Arora

Dep. of Mathematics & Statistics, GKV, Hardwar

This talk aims at presenting a brief overview of mathematical activity in Vedic literature.

6
NEW DIRECTION IN VEDIC MATHEMATICS

S. L. Singh

Dep. of Mathematics & Statistics, GKV, Hardwar

The purpose of this talk is to identify some new components of mathematics of Vedic tradition, which has relevance with modern taste in mathematical sciences.

7
SIMILIRITY OF PLANE FIGURE AND GEOMETRIC AND GROUP THEORETIC STUDY OF CYCLIC QUADRILATERAL OF FIRST KIND

Vinod Mishra and S. L. Singh

Dep. of Maths, Sant Longawal Inst. of Engg. & Tech., Longowal, Punjab.
Dep. of Mathematics & Statistics, GKV, Hardwar

Similarity of plane figure is a very old concept. The approach is to study the valuable contributions by the Arabs, Babylons/Summerians, Chinese and Greeks with reference to the Indians at large and importance of the concept as such. Finally, we study cyclic quadrilateral of first kind geometrically. Its group theoretic study concludes with a non-abelian group of eight permutations in which four permutations are equal to another four.

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8
SOME INDIAN SAVANTS OF MATHEMATICS

Man Mohan and Subhash Chander

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Thousands of years ago, Indian mathematicians authored various theses and dissertations on mathematics. It is now commonly believed and widely accepted that these treatises laid down the foundations of algebra, algorithm, square roots, cube roots, various methods of calculation, and the concept of zero. It is also believed, that in Vedic times mathematical formulae were often taught within the context of spiritual expression (mantra) and thus while learning spiritual lessons, one could also learn mathematics. Here we present a brief account of 38 Indian savants of Mathematics covering the period 800 BC–1980 AD. Most of the information is based on historical resources available on the Web and arranged in the chronological order of their known or guessed periods.

9
A BRIEF SURVEY OF ANCIENT MATHEMATICS – A SHORT NOTE ON COMMERCIAL MATHETMATICS

J. Achari

Dep. of Mathematics, Statistics and Computer Science
NES, Science College Nanded (Maharastra)

10
GOLDEN PEROID OF INDIAN MATHEMATICS

B. B. Kulkarni

Netaji Subhas Chandra Bose College, Nanded (Maharastra)

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11
THE KRIYAKRAMAKARI EXPOSITION ON MADHAVA SERIES APPROXIMATIONS

Madhukar Mallayya

Chandni, TC 25/1975, Deshabhimani Road, Trivandram (Kerala)

The Kriyakramakari is a sixteenth century commentary in Sanskrit on the famous Lilavati of Bhaskaracarya. Commenting on Bhaskara's enunciation for determination of gross and nearly precise circumference (suksma paridhi and sthula paridhi) of a circle, the commentator Sankara Varrier gives an elaborate and excellent exposition on this topic by incorporating several methods for getting better and better approximations attributed to his ancestors. Tremendous achievements made by Sangamagrama Madhava of fourteenth century in the field of infinite series and series approximations are highlighted in this section of the commentary. The commentator refers to various methods based on infinite series and gives an excellent demonstration of them by employing certain ingenious techniques of modern analysis. Sankara gives an interesting derivation of Madhava's correction functions for approximating certain series. Moreover, various other infinite series are found to have been deduced from these correction functions.

12
INNOVATIONS IN KERALA ASTRONOMY

S. Madhavan

40/513, 3rd Puthen St, Trivandrum

In the post Bhaskara period, Kerala has been a centre of stormy developments in mathematics and astronomy. Some of the innovations are described here. First of all, the concept of Jya and the table of Jyas introduced by Aryabhata, Varahamihira and others underwent drastic changes. Madhava found the series for PI and changed the figure 3438' for the radius or trijya as 3437' 44" 48"'. He also found the infinite series for R sine and R cosine. The proof given in Yuktibhasa abounds in novel concepts.

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Madhava's amendment of Chandravakyas and his interesting method of finding the moon's position using the properties of periodic functions are strikingly original. The methods of successive approximation used in various contexts also deserve investigation. Some results in Karapaddhathi and Puthumana Somayaji and Satratnamala of Sankaravarman indicate good knowledge of spherical trigonometry.

13
MEASURE THEORY – HISTORICAL PERSPECTIVE

Pramila Srivastava

Allahabad Mathematical Society, 10, C.S.P. Singh Marg, Allahabad-211 001

The purpose of this talk is to present a development and discuss its evolution.

14
ALGEBRA OF NĀRĀYAṆA

S. L. Singh and M. M. Dwivedi

Dep. of Mathematics and Statistics, GKV, Hardwar
Mahamati Pran Nath Mahavidyalaya, Chitrakoot

The first author has recently found Gaṇitavatansa (c.1356. A.D.) of Nārāyaṇa Paṇḍita from Mr. Vikas Ghosal of West Bengal. The authors have under taken the study of this rare text-book of algebra, which is hopefully akin to the algebra of Bhāskara II. The purpose of this talk is to present some glimpses from the same.

15
RELEVANCE OF HISTORY IN MATHEMATICS TEACHING

J. M. C. Joshi

Kumaun University, Nanital, 47 A Heera Nagar, Haldwani

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16
PLACE VALUE SYSTEM IN TRADITIONAL ALGEBRA

Ramesh Chand

Government Sarvodya Bal Vidyalaya, B-Block, Nand Nagri, Delhi

In this paper an attempt is made to discuss the concept of place value system in algebra.

17
FRACTALS IN LĪLĀVATĪ

Mamta Rani

Dep. of Computer Sc. & Engg., SRM CEM, Faizabad Road, Lucknow

Fractal theory, a newly developed branch of computer graphics is based on feedback system. However, this field has been recognized three decades back but its germs are found in mathematics of Vedic tradition. One of the classical fractals has been found in Bhāskarācārya's Līlāvatī known as *Meru*, generally called Pascal triangle. Some recently developed fractals are also akin to figures found in ancient Indian compositions.

18
THE PENDULA IN MATHEMATICAL SCIENCES

R. S. Kaushal

Dep. of Physics & Astrophysics, University of Delhi

The role played by the pendulum and its variants in the development of mathematical sciences in reviewed. In particular, the case of a pendulum, viz., harmonic oscillator in physics is highlighted in the context of mathematical, conceptual, conventional and engineering disciplines beside the one in mathematical physics. It is pointed out that in all these applications the concept of pendulum is used as a vehicle of knowledge and mainly on the basis of structural

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analogy. Finally, its role in an example of the consciousness-manifesting phenomenon will be discussed.

19
SOME BASIC ASPECTS OF COMPUTER TECHNOLOGY IN VEDIC LITERATURE

Vinod Kumar

Department of Computer Science, GKV, Hardwar

Vedas are considered to be the first written book of the world and a treasure of knowledge containing all kinds of knowledge in the world in its basic form. Computer technology consists of hardware, software and humanware. Computer hardware basically depends upon the controlled flow of electricity keeping its level very low (-12V to 12V). Several mantras of Rigveda mentions about the advantage of controlled flow of electricity that provides the basis for hardware design of computers. Mantras, concerning the use of air and electricity for massage transmission, are also available in Vedic literature. Also, the Sutras of Vedic mathematics like "Ekadhiken Poorvena" have tremendous power of algorithmic representation.

The present paper aims at exploring the roots of electron flow, microwave transmission, computer networking and algorithmic representation of solution process in the Vedic literature.

20
HISTORY OF DEVELOPMENT OF BINOMIAL EXPANSIONS

V. K. Sharma and Anita Sharma

Dep. of Mathematics and Statistics, GKV, Hardwar

In this paper contribution of various mathematicians like Varahmihir, Mahaviracharya, Bhaskaracharya, Umar Khayyam, Pascal, Newton and Gauss etc. in development of Binomial expansion and its validity is discussed.

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21
A DEVELOPEMENT OF APPROXIMATION THEORY IN INDIA — A CRITICAL REVIEW

S. P. Singh

Dep. of Mathematics, Guru Ghasi Das University, Bilaspur (Chhattisgarh)

22
FIXED POINTS FOR SET-VALUED MAPS

Devanand Ram, P. P. Murty and S. N. Lal

Dep. of Mathematics, Faculty of Science, BHU, Varanasi.

It is well known that Banach Contraction principle has been extended and generalised in several different forms by various authors. In this paper we have established a common fixed point theorem considering the Greguš Φ-contraction in the setting of set-valued mappings.

23
ALGEBRA FOR COMPUTERS

P. Pradhan and Girish Sharma

Dept. of Mathematics, GKV, Hardwar
Dep. of Computer Science, Hindu P.G. College, Sonipat (Haryana)

This paper is presenting the truth & validity of the prime (simple) & compound statement using connectives and attempts are made to establish the rule of modern algebra by implementing in terms of Boolean Algebra. Switching circuits are also discussed along with some digital circuits as an application of Boolean Algebra.

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24
FIXED POINT THEOREMS FOR MEIR-KEELER TYPE HYBRID CONTRACTION

S. L. Singh and Anita Tomar

Dep. of Mathematics, GKV, Hardwar

Beg and Azam initiated the study of fixed points for a pair of single-valued and multivalued maps satisfying Meir-Keeler type contraction condition. The intent of this paper is to obtain fixed point theorems for a Meir-Keeler type hybrid contractions, which extend and improve main results.

25
TRANSFORMATION OF GEOMETRICAL FIGURES

Nidhi Handa

Dep. of Mathematics & Statistics, KGM, GKV, Hardwar

This paper aims at the conversion of area i.e. the conversion from rectangle into square and vice-versa, etc. It focuses on how the four-sided complicated structures have been simplified in the sulb-sutra. Here the endeavour has been also made to highlight the method of construction and proof of each sutra.

26
USEFUL WEB LINKS ON HISTORY OF MATHEMATICAL SCIENCES

Man Mohan

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The Internet is a wonderful place to explore the vast amount of information that it makes available around the world, but without borders and often confusing the explorer. Even well prepared surfers stumble aimlessly through cyberspace using

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hit-or-miss methods in search of useful information with few results, little substance and lots of frustration. The purpose of this article is to encourage and facilitate the teaching and learning of history of mathematics at all levels, using the Web. Some of the most useful resources on the history of mathematics are identified, described, and hyperlinked. The resources are divided into twelve sections. While some websites may fall in more than one section, each website appears in what seemed to be the most relevant section.

27
EFFICIENT FAMILIES OF CHAIN ESTIMATORS FOR RATIO AND PRODUCT OF TWO POPULATION MEANS

H. S. Jhajj, M. K. Sharma and Lovleen Kumar Grover

Guru Nanak Dev University, Amritsar

Here we proposed efficient families of chain estimators for estimating ratio and product of two population means using known information on one of variable of the given two auxiliary variables. The bias and mean squared error of proposed families are obtained up to first approximation. The optimum mean square errors are also obtained. The gain in efficiency is illustrated empirically.

28
REDUNDANT SYSTEM WITH N-REPAIRABLE SERVERS

Charanjeet Singh and Madhu Jain

Dep. of Mathematics, Guru Nanak Dev University, Amritsar
Dep. of Mathematics, Institute of Basic Sciences Khandari, Agra

This investigation deals with steady state availability analysis with r repairmen where k_i-n_i- out system fails, if n_i-k_i units fail in i^{th} server. The Hessenberg matrix determinants approach in recursive manner along with Markov chain balance equations has been provided to determine the availability analysis. The parallel - series replicated technique has been adopted for illustration of steady state availability.

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29
ROLE OF DYNIMICAL SYSTEMS AND ITS IMPLICATIONS

R. P. Pant and K. Jha

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E-mail:jhaknh@yahoo.co.in

The basic goal of the theory of dynamical systems is to understand the eventual or asymptotic behaviour of an iterative process. An important notion in the study of the dynamical systems is the stability or persistence of the system under small changes or perturbations, which is the concept of structural stability and the concept of structural stability is extremely important in applications of the theory of the dynamical systems. Dynamical systems occur in all branches of science, from the differential equations of classical mechanics in physics to the difference equations of mathematical economics and biology. The last thirty-five years have seen an explosion of interest in the study of the non-linear dynamical systems. Scientists in all disciplines have come to realize the power and the beauty of the geometric and qualitative techniques developed in the dynamical systems. The aim of the present paper is to give a brief account on the history, important results and some applications of dynamical systems.

30
QUASI-CONTRACTIONS AND APPROXIMATE FIXED POINTS

S. L. Singh and Bhagwati Prasad Chamola

Department of Mathematics, GKV, Hardwar

Approximate fixed points of the Banach contraction on non complete metric spaces exist. In this note, we discuss approximate fixed points of quasi-contractions on metric spaces.

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31
PERFORMANCE EVALUATION OF COMPUTER COMUNICATION SYSTEMS

Manisha Sharma, P. K. Yadav and Avanish Kumar

Bansal Institute of Engg. & Tech. Bhopal, C.B.R.I., Roorkee
Bundelkhand University, Jhansi

A Computer Communication System consists of a set of processing nodes, interconnected by to some extent by communication links. In a non-redundant system, each processing node contains exactly one processor and there is at most one link between any pair of processing nodes. The present paper discusses a general queue model to the system represent transmitter and receiver connected through communication channel. The mean response time, based on mean waiting time, mean transmitting time and mean service time is than obtained to determine the performance of the systems.

32
OPERATION RESEARCH - A HISTORICAL NOTE

M. P. Singh, P. K. Yadav and Avanish Kumar

G.K.V. Hardwar, C.B.R.I., Roorkee,
Bundelkhand University, Jhansi

The Operation Research is one of the important disciplines of the Mathematical Sciences, which deals a large number of problems related to Management, Engineering, Industry, Bio-Sciences etc. The role of these techniques is involving the day-to-day matters of the human beings. It suggests how to best use of the resources in lucid manner. The present paper focuses on the various stages of the development of the Operation Research and its concern ones.

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33
HISTORY AND DEVLOPMENT OF BANACH CONTRACTION PRINCIPLE

U. C. Gairola

Dep. of Mathematics, Pauri Campus of HNB Garhwal University, Pauri

Using the concept of convergence of a cauchy sequence in a normed linear space S. Banach gave an important, concept of a special space which is named as BANACH spaces later on. This space plays a vital role in applicable Mathematics. The most and the first important application of Banach concept is used by himself by introducing contraction principle in metric spaces. In this paper I shall discuss the history, development and some basic applications of Banach contraction principle.

34
PERMUTATION AND COMBINATION WITH ITS BRIEF HISTORY IN ANCIENT MATHEMATICS

Naubahar Singh

VMIC Sector-1, BHEL, Hardwar

In this paper an attempt is made to sketch a brief account of its history and discuss a few glimpses of Bhaskaracharya's II, techniques in treating the problems concerning to permutation and combination.

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35
STABILITY OF MANN AND ISHIKAWA ITERATIVE PROCEDURES FOR CONTRACTIVE MAPS

S. L. Singh and Charu Bhatnagar

Department of Mathematics, GKV, Hardwar

Let X be a Banach spaces and T a self map of X. Let x_0 be a point of X and $x_{n+1} = Tx_n$ denotes an iteration procedure which yields a sequence of points $\{x_n\}$. Suppose that $\{x_n\}$ converges to a fixed point p of T. Let $\{y_n\}$ denote an arbitrary sequence in X and $\varepsilon_n = \|y_{n+1} - Ty_n\|$. If $\lim n\ \varepsilon_n = 0$ implies that $\lim_n y_n = p$ then the iteration procedure is to be T-stable. Let T:X->X , then there exists a constant satisfying $0 \leq c < 1$ such that , for each $x,y \leq X$, $\| Tx-Ty \| \leq cM(x,y)$
Where, $M(x,y) = \max[\| x-y \|,\| x-Tx \|,\| y-Ty \|,1/2\{\| x-Ty \|+\| y-Tx \| \}]$
In this paper, we show that several interval iteration procedures are stable for maps T satisfying above condition.

36
SOME FIXED POINT THEOREMS ON 2-METRIC SPACES

D. D. Sharma

Department of Mathematics, GKV, Hardwar.

This paper establishes some fixed point theorem for Matkowski-type contraction mappings defined on 2-metric spaces.

37
ANALYSIS OF STRESS- INTENSITY FACTOR FOR AN EPITROCHOIDALDISC UNDER TWO CONCENTRATED FORCES

Rasid Ali, M. P. Singh and Parul Rani

Dep. of Mathematics, Vishveswarya Institute of Engg. & Tech. Dadri, G.B.Nagar
Dep. of Mathematics and Statistics, G KV, Hardwar

Complex variable technique has been found to be very effective for solving plane, axisymmetric and boundary value problems in Solid Mechanics. Employing this

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method, the first fundamental plane problems of an epitrochoidal disc has been discussed. It is assumed that the plate is thin, isotropic and homogeneous under two concentrated forces applied at the extremities of its major axis of the exterior bounded epitrochoid. Stresses are found out and the variation of stress-intensity factor has been studied graphically. The solution of an elastic circular plate and that of a plate in the form of a cardioid has been obtained as particular cases with same boundary conditions.

38
MAGNETIC TORQUE EFFECT ON SATELLITE MOTION

Parsan Kaur and Rashmi Bhardwaj

Dep. of Mathematics, School of Basic & Applied Sciences
Guru Gobind singh Indraprastha University, Kashmere Gate, Delhi 110 006

Magnetic disturbance torque results from the interaction between the spacecraft's residual magnetic field and the geomagnetic field. The primary sources of magnetic disturbance torques are (i) spacecraft magnetic moments, (ii) eddy currents and (iii) hysteresis. Of these, the spacecraft's magnetic moment is usually the dominant source of disturbance torques. Let us consider a rigid satellite S moving around the Central Body C. We further suppose that the orbital plane of satellite coincides with the equatorial plane of the central body. We discussed that the non-linear rotational equations of motion of the planar oscillation of a satellite in an elliptic orbit under the influence of magnetic torque are non-integrable. Also we have observed graphically that in the Earth-Moon system, the Melnikov's function has a simple zero and hence the equations of motions are non integrable.

39
AN APPROCH TO NEW TRICORNS

S. L. Singh And Ashish Negi

G.K.V. Hardwar, Govind Ballabh Pant Engg. College, Pauri Garhwal

Recently using Mann iterative procedure, Kumar [3] (see also Rani [4]) have introduced superior Julia sets for polynomials. In this paper following the same techniques we obtain new tricorns arising from quadratic polynomials.

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40
EFFECT OF DAMPING ON VIBRATIONS OF AN ISOTROPIC NON-HOMOGENENOUS CIRCULAR PLATE OF PARABOLICALLY VARYING THICKNESS

Rajendra Kumar, Virendra Arora and Manu Gupta

Dep .of Mathematics, J. V. Jain College, Saharanpur
Dep. of Mathematics and Statistics, GKV, Hardwar

The research in the field of vibration is unceasingly acquiring immense importance in modern science as it has in every field of applied science today. The effect of vibration on design of machine, multistoried building, radio-telescope, nuclear reactor technology, earthquake resistant structure and various other engineering structures is of immense importance. Today we are going ahead in space technology where problems of plates continuously supported by elastic and visco-elastic media are considered. Also effect of damping is applied for practical use.

This paper deals with the free vibrations of damped non-homogeneous isotropic elastic circular plate of parabolically varying thickness. The modulus of elasticity 'E' and density 'ρ' of the plate material are assumed to vary linearly with the radius vector. The transverse displacement is expressed as a power series in terms of the radial co-ordinates and Frobenius method is used to solve the governing differential equation of motion. Frequency equations have been derived for the two combinations of boundary condition namely clamed –simply supported and clamped plate. The numerical results corresponding to the first two modes of vibrations have been computed for different values of taper constant, non-homogeneity and damping parameter.

41
A STUDY OF ANCIENT ASTRONOMICAL INSTRUMENTS AND THEIR RELEVANCE TO MODERN TIME

V. K. Sharma and Mustafizur Rahman

Dep. of Mathematics and Statistics, GKV, Hardwar

This paper gives a concise account of ancient and modern astronomical instrument with the development of astronomical science in India. On the other

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hand this account has been placed in the context of historical developments of instrument in India. In addition to ancient instrument the brief account of contributions by *Brahmagupta* and Raja Jai Singh is also included.

42
NUMBERS IN VEDIC LITERATURE

Savitri Verma and Shiva Mittal

Dep. of Mathematics, Dr. S. P. M. Govt. Degree College, Allahabad

This paper intends to survey theory of numbers in Vedic literature.

43
A DEVELOPMENT OF THE SCIENCE OF COMBINATORICS IN ANCIENT INDIA

Parmeshwar Jha and N. K. Sinha

Saraswati Sadan and Vidyapuri, Supaul (Bihar)
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Importance of the topic 'combinatories' in mathematics cannot be over emphasised. We have become so familiar with the subject that we seldom feel necessary to think how and when it was invented. Consequently very little efforts have been made in the past to record its gradual evolution especially in ancient India. The present paper is an attempt in this direction. There are grounds to show that its origin can be traced back to the period of Vedas. Principles and techniques for the solution of various types of combinatorial problems are also found to exist in the Sushrutn Samhita, Pingala's Chandah Sutra, Jaina Canonieal works and other literature of ancient India.Moreever, Inter Indian Scholars, viz., Mahavirracharya, Halayudha, Bhaskaracharyn, Narayana Pandita and others have also explained several techniques in their works. On the whole, remarkable progress in the field has been made in different periods . The paper discusses all these achievement in detail and presents a chronological account of the growth and development of the science of combinatorics in ancient India.

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44
MHD UNSTEADY FLOW OF A DUSTY VISCO-ELASTIC FLUID INDUCED BY THE MOTION OF A SEMI-INFINITE PLATE MOVING WITH VELOCITY DECREASING EXPONENTIALLY WITH TIME

B. S. Rawat

Department of Mathematics, HNB Garhwal University, Srinagar

In this paper, we have considered the velocities of dust particle and dusty fluid are obtained in presence of magnetic fields for the flow induced in an incompressible dusty fluid by the motion of semi-infinite flat plate moving with velocity decreasing exponentially with time. The velocity distribution of dusty fluid, dust particle and distribution of magnetic field in case of dusty fluid are represented graphically. Effect of elastic parameter, Hartmann number on the velocity and magnetic profile are also discussed.

45
A COMPARISON OF BANACH AND KANNAN FIXED POINT THEOREMS

S. L. Singh, Anurag Naithani, R.P. Pant

Dep. of Mathematics & Statistics, GKV, Hardwar.

46
COMMON FIXED POINT THEOREMS FOR FAMILY OF CONTRACTIVE TYPE MAPPINGS

S. L. Singh and Ashish Kumar

Dep. of Mathematics and Statistics, GKV, Hardwar

Common fixed point theorems have been proved for a family of mappings satisfying a minimal type commutative and contractive conditions. A fixed point theorem using E.A. property is also obtained.

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47
ON COMMON FIXED POINT THEOREMS IN D-METRIC SPACES

Virendra Arora and Mohini Pundir

Dep. of Mathematics and Statistics, GKV, Hardwar

In this paper, fixed point theorems for a weakly compatible mappings in D-metric spaces satisfying certain strict contractive conditions are proved which include the fixed point theorems of Dhage as the special cases under weaker conditions.

48
NEW APPROACH TO GRAPHICAL MODELING

S. L. Singh and Manish Kumar

Dep. of Mathematics & Statistics, GKV, Hardwar,
Dep. of Comp. Sc. & Engg., SRM CEM, Faizabad Road, Lucknow

Fractal Theory is a newly developed branch of Computer Graphics, based on feedback system. Images can be generated not having definite structures. Fractals are everywhere, e.g., medical science, wavelet theory, telecommunications, image compression and several other fields. This technique is simple to understand and easy to compute to solve problems, which yields realistic models.

49
DIFFERENT TYPES OF FUZZY MEASURES AND ITS APPLICATIONS IN DECISION THEORY

R. C. Dimri

Dep. of Mathematics, H.N.B. Garhwal University, Srinagar

Fuzzy set theory has formulated different types of uncertainty, which were before either not recognized properly or considered under probability theory. Various special classes of measures can be generated by the concept of fuzzy measure. Fuzziness can be introduced at several points in the presently existing models of

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decision making. This contribution investigates some class of fuzzy measures and relationship among them.

50
AN IDENTITY INVOLVING GOLDEN RATIO –A HISTORICAL PERSPECTIVE

Jamuna Prasad Ambasht

34 Westpine Court, Columbia, S.C. 29212 (U.S.A.)

The purpose of this talk is to make a modern mathematician travel through the lanes of Channai and acquaint oneself with the days of Ramanujan's survival amidst all hardships. The case in point is the nested square-roots, as also the continued fractions.

If one can recall the letters from an Indian Clerk to an English Professor, one could not afford to miss that part wherein Ramanujan claims Beta and Gamma functions involving integration lead to elegant identities. Hardy had no difficulty in understanding this. What threw him off was another assertion using the fifth roots of x in the numerators and the denominators of continued fractions.

The claim we are making is that a golden ratio g satisfies $g^2 = g+1$ and therefore turns out to be u and v of Ramanujan and therefore fitsinto his identity involving powers of u and v. (see last equation of Ramanujan's letter to Hardy). This leads to beautiful new results in case of nested square-roots and also continued fractions as we can always express g in these forms.

51
MATHEMATICAL ENGLISH

Asim Majumdar and Sandeep Bhakat

Siksha-Satra, Visva Bharati, Sriniketan

In our country where different provincial languages are spoken English plays a very significant role as a means of communication. The objectives of the project are to teach English language to the first generation learners particularly for their English literacy by developing a new type of method for English language

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learning. Some mathematical tools and new concepts can justify the methodology. The project also aims at developing research-based techniques for easy and scientific English learning. The project also undertakes to prepare a wordbook designed with the words widely used in learning of English language in its basic level. The project attempts at characterizing the words in terms of usage in English language. Whether the reading materials developed in this project are very effective and scientific can be verified with the help of mathematical sciences particularly by the application of Fuzzy set theory to evaluate the potentiality of a word of different order.

As for example: Consider the set of all English words consisting of 3 letters with 'N' in the first position. There are 26x26 numbers of such combinations out of which 17 are considered English words. The problem of evaluating the potential value of one such word with respect to its frequency of use in communicating English. This potential value can be derived by the application of fuzzy set theory.

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AIR POLLUTION AND ITS EFFECTS ON ATMOPSPHEREIC PROPERTIES

Virendra Arora and Ajendra Kumar

Dep. of Mathematics and Statistics, GKV, Hardwar.

Air pollution has posed a serious problem for our atmospheric properties. In this paper air pollution and its effects on human being, vegetation, materials etc. has been discussed.

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EVOLUTION AND DEVELOPMENT OF NUMBER SYSTEM

N. A. Pande

Dep. of Mathematics, NES College, Nanded (Maharastra)

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APPROXIMATE FIXED POINT THEOREMS

Abha Singh
Bela, Via Narayanpur, Mirzapur(U. P.)

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WEAKLY TIGHT FUNCTIONS AND THEIR DECOMPOSITION

Pramila Srivastava, Mona Khare and Bhawna Singh

Allahabad Mathematical Society, 10, C.S.P. Singh Marg, Allahabad, India
Department of Mathematics, University of Allahabad, Allahabad

The present paper deals with the study of a weakly tight function and its relation to tight and cotight functions. We have proved that every tight function is weakly tight and also every cotight function is weakly tight. We have obtained that a Jordan decomposition type theorem for a locally bounded function which is weakly tight followed by the notion of total variation .In addition, it is proved that if β is additive and modular, then the decomposed parts $\beta+$ and $\beta-$ are super additive and super modular.

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DOUBLE INDUCTION

C. S. Bisht

Department of Mathematics, KumaunUniversity, Nainital

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D. L. Despande

Dep. of Mathematics, NES College, Nanded (Maharastra)

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RULES IN VEDIC MATHEMATICS AND PROGRAMMING

S. L. Singh and Nilam Aswal

Dep. of Mathematics and Statistics, GKV, Hardwar

Computer programming based on certain rules of Vedic mathematics are discussed.

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PICARD ITERATIVE PROCEDURE IN ĀRYABHAṬIYAM

P. Pradhan

Dep. of Mathematics and Statistics, GKV, Hardwar

Iterative procedures akin to Picard iterations are used by Āryabhaṭa (b. 499 A.D.) and later mathematicians of Vedic traditions. This paper intends to discuss this procedure and related problems.

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ARYABHATT – HIS CONTRIBUTIONS TO MATHEMATICS

D. S. Hooda

Kurukshetra University, Kurukshetra

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APPLICATION OF LIAPUMOR FUNCTION TECHNIQUE IN THE STUDY OF ENTRY PROFILE OF A SPACE VEHICLE

B. Ram

Dep. of Mathematics, Tehri Campus of HNB Garhwal University, Tehri Garhwal

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APPLICABLE MATHEMAICS: AN OVERVIEW

V. K. Katiyar

Dep. of Mathematics, IIT, Roorkee

Due to increasing demand of mathematical techniques in developing analytical and experimental models in all branches of science and technology for industrial development of the country, an overview will be presented as an application of mathematics in Biological, Social, Economic and in industrial systems.

63
A DEVLOPMENT OF HYBRID CONTRACTIONS AND COINCIDENCES

S. L. Singh and Amal M. Hashim

Gurukula Kangri Vishwavidyalaya, Hardwar

Study of hybrid contractions on metric spaces was initiated by S. L. Singh, Chitra Kulshrestha, R. Bhaskran, P. V. Subrahmanyam and Olga Hadzic during 1980-1984. Thereafter a number of papers have appeared in this line. The purpose of this talk is to present a brief historical development of hybrid contractions, and we also present some new coincident theorems for very general type of hybrid contractions have E.A. property.

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संस्कृत वाङ्मय एवं अंक विद्या

जयकृष्ण गोदियाल

हे0 नं0 ब0 गढ़वाल विश्वविद्यालय, पौड़ी परिसर
पौड़ी 246001 उत्तरांचल

संस्कृत वाङ्मय संसार में सर्वाधिक प्राचीन एवं उपयोगी है। इस भाषा के साहित्य में विषयों की विविधता इसकी विशेषता है। साहित्यिक विषयों के साथ-साथ वैज्ञानिक विषयों के प्रतिपादन की इसकी प्रवृति के कारण ही सम्पूर्ण विश्व जनमानस अत्याधुनिक इस युग में भी संस्कृत वाङ्मय के अध्ययन की ओर गहराई से उन्मुख है। वैदिक वाङ्मय वस्तुतः विज्ञान का भंडार है। संस्कृत वाङ्मय में यत्र-तत्र पड़े गणितीय विचारों को उद्घाटित करना ही इस प्रस्तौष्यअण शोध पत्र पर मुख्य विवेच्य विषय है।

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भारत में कूटविज्ञान

सुन्दर लाल

गणित विभाग
डा0 भीमराव अम्बेदकर विश्वविद्यालय, आगरा

भारत में कूटविज्ञान की प्राचीन परम्परा है। इस प्रपत्र में इन परम्पराओं का कूट लेखन एवं उसके प्रेषण के सम्बन्ध में प्रकाश डाला गया है।

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NATIONAL SEMINAR
ON
HISTORY OF MATHEMATICAL SCIENCES
&
APPLICABLE MATHEMATICS

Organised by

DEPARTMENT OF MATHEMATICS AND STATISTICS

GURUKULA KANGRI VISHWAVIDHYALAYA, HARDWAR

DAILY PROGRAMME

MARCH 07, 2003

VENUE: Conference Hall, Faculty of Management

09.00 REGISTRATION

10.00 Yajna by Prof. Bharat Bhushan & Acharya Prof. Ved Prakash, Pro VC

10.30 INAUGURAL SESSION

Welcome Address by Prof. S. L. Singh
Organizing Secretary & Principal, Science College

Address by Prof. Mahavir Agrawal, Registrar

Address by Acharya Prof. Ved Prakash, Pro VC

Inaugural Address by Chief Guest Shri Ramphal Bansal
Senior Advocate, Delhi High Court

Address by Prof. B. S. Yadav
Chairman, Organizing Committee

Key Note Address by Prof. G. S. Pandey

Presidential Address by Prof. Swatantra Kumar, Vice-Chancellor

Vote of thanks by Prof. Virendra Arora
Organizing Secretary & Dean Faculty of Science

11.30 TEA

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12.00 - 12.45	Yamataization Prof. B. S. Yadav, Delhi Chairperson: Prof. G. S. Pandey
12.45-13.15	Similairity of plane figure & geometric & group... Dr. Vinod Mishra, SLIET, Longowal Chairperson: Dr. Kanhaiya Jha
13.15	LUNCH
15.00-15.30	Applicable Mathematics : An overview Prof. V. K. Katiyar, IIT, Roorkee Chairperson: Prof. S. P. Singh
15.30-16.00	Useful Web Links on History of Mathematical Sciences Dr. Man Mohan, Delhi Chairperson: A. B. Lohni
16.00-1630	Inauguration of Exhibition on Vedic Sciences followed by Tea
16.30-17.30	TALKS/PAPER READING SESSION **Talks/papers by:** Prof. J. K. Godiyal, Amal M. Hashim, Asim Majumdar, Sandeep Bhakat, N. A. Pandey, D. L. Despande, Nilam Aswal, P. Pradhan, B. Ram, B. S. Rawat, Ashish Kumar, Mohini Pundir Chairperson: Dr. Man Mohan
17.30-19.30	Visit to Har -ki- Pauri
19.45-20.30	DINNER

VENUE: Faculty of Science

March 08, 2003

09.30-10.00	Measure Theory-Historical Perspective Prof. Pramila Srivastava, Allahabad Mathematical Society, Allahabad Chairperson : Prof. B. S. Yadav
10.00-10.20	New Directions in Vedic Mathematics Prof. S. L. Singh, Hardwar Chairperson : Prof. J. M. C. Joshi

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10.20-10.40	A Brief Survey of Ancient Mathematics ... Dr. J. Achari, Nanded Chariperson : Prof. Virendra Arora
10.40-11.10	The Kriyakramakari Exposition on... Dr. Madhukar Mallayya, Trivandram Chariperson: Dr. J. Achari
11.10-11.25	TEA
11.25- 11.45	Innovations in Kerala Astronomy Dr. S. Madhavan, Trivandrum Chairperson: Prof. R. S. Kaushal
11.45-12.15	The Pendula in Mathematical Sciences Prof. R. S. Kaushal, University of Delhi Chairperson: Prof. Vinod Kumar
12.15-12.40	Double Induction Dr. C. S. Bisht, Kumaun University, Nainital Chairperson : Dr. Madhukar Mallayya
12.40-13.00	Fractals in Lilavati Dr. Mamta Rani, Lucknow Chairperson : Dr. S. Madhavan
13.00-14.15	LUNCH
14.15-17.00	Excursion to Rishikesh including a visit to Paramarth Niketan
15.00-15.30	Executive Council Metting, ISHM
17.00-19.00	Shri Ganga Aarti, Blessings by Svami Chidanand Sarsvati "Muni Ji" and Prasadam

March 09, 2003

19.30.10.00	Innovations in Kerala Astronomy Dr. S. Madhavan, Trivandrum Chairperson: Prof. C. S. Bisht
10.00-10.20	Algebra of Narayana Dr. M.M. Dwivedi, Chitrakoot Chairperson: Prof. M. P. Singh

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10.20-10.40	Relevance of History in Mathematics Teaching Prof. J. M. C. Joshi, Nainital Chairperson: Prof. Pramila Srivastava
10.40-11.00	A developement of Approximation Theory... Prof. S. P. Singh, Guru Ghasi Das University, Bilaspur Chairperson: Dr. S. Madhavan
11.00-11.15	TEA
11.15-11.35	Mahavira;s Method of False Position Dr. Kanahiya Jha, Jamshedpur Chairperson: Prof. V. K. Sharma
11.35-11.55	The Pendula in Mathematical... Prof. R. S. Kaushal, Delhi Chairperson: Prof. G. S. Pandey
11.55-13.20	**Talks/ paper by:** Subhash Chander, B. B. Kulkarni, Ramesh Chand, Anita Sharma, Devanand Ram, P. Pradhan, Grish Sharma, Anita Tomar, Nidhi Handa, H. S. Jhajj, M. K. Sharma, L. K. Grover, Charanjeet Singh, K. Jha, B. P. Chamola, M. Sharma, P. K. Yadav, A. Kumar, U. C. Gairola, N. Singh, C. Bhatnagar, D. D. Sharma, P. Kaur, R. Bhardwaj, A. Negi, M. Gupta, M. Rahman, S. Verma, M. Kumar, R. C. Dimri. Chairpersons: Prof. S. P. Singh and Prof. B. Ram
13.20-13.40	**Valedictory Function**
13.40	LUNCH

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Vol. 17 No. 2, 2003

प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका-

JOURNAL OF NATURAL & PHYSICAL SCIENCES

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Journal of Natural & Physical Sciences

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Journal of Natural & Physical Sciences, Vol. 17 No. 2 (2003) 57-76

STABILITY AND STABILITY SWITCHING IN 3-SPECIES MODEL WITH DELAYS

IDRISH H. ELMABRUK* AND H. L. ARORA*

(Received 02.01.2002)

ABSTRACT

Sufficient conditions for the stability of the positive equilibrium of a 3-species Lotka-Volterra-type system with density-dependent death rates modeling the interactions of two competing predators living exclusively on a common prey with delays, are obtained by constructing a suitable lyapunov function. An estimation of delay length for which equilibrium stability holds is obtained. We show the existence of some critical value $\hat{\sigma}$ of the delay σ in the predation and biomass conversion terms at which the stability switches to instability and the system remains unstable for $\sigma > \hat{\sigma}$.

Key words: Predator-Prey System, Equilibrium, Lyapunov Function, Delay, Stability-Switching.

INTRODUCTION

One of the most interesting questions in ecological models concerns with local stability of the positive equilibrium of the model. The question becomes even more interesting when we consider time delays in the growth dynamics of the interacting populations of various species. Delays can arise from a great variety of causes and undoubtedly always present to some extent. In order to, realistically, model a real life system of inter-specific (and intra-specific) interations, one generally needs to take into account the influence due to the past states of such a system, or the waiting period resulting because of physiological processes. These usually mean that a real system should be modeled by differential with time delays. Indeed, such a view is gaining popularity across the board of science and engineering, as evidenced by the rapid increase in the number of mathematical models of delay differential equations studied in various scientific disciplines, e.g., see the numerous references cited in [1]. This is specially true in the area of mathematical biology, see [4,8,11].

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Considering the historical Lotka-Volterra 2-dim system without delays, that is

$$\begin{aligned}\frac{dH}{dt} &= aH - bHP \\ \frac{dP}{dt} &= -dH + cHP\end{aligned} \tag{1.1}$$

where a, b, c, d are positive constants, it is very well known that (1.1) is unrealistic for various reasons. We, for simplicity, nondimensionalize (1.1) by

putting $u(\tau) = \frac{cH(t)}{d}, v(\tau) = \frac{bP(t)}{a}, \tau = at, a = \frac{d}{a}$, then (1.1) becomes:

$$\begin{aligned}\frac{du}{d\tau} &= u(1 - v) \\ \frac{dv}{d\tau} &= \alpha v(u - 1)\end{aligned} \tag{1.2}$$

The trajectories of (1.2) are given by

$$\alpha u + v - v\ln(u^{\alpha}) = N, \text{where} N > 1 + \alpha, \tag{1.3}$$

and in the phase plane are closed (periodic solutions) as shown in Fig. (1.1), a typical periodic solution $(u(\tau), v(\tau))$ is shown in Fig. (1.2). A major inadequacy

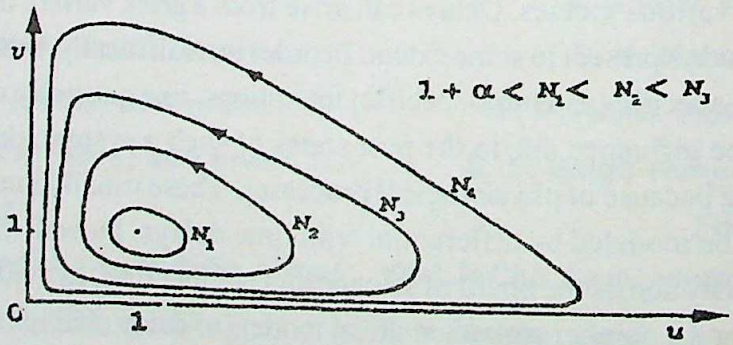

Fig. (1.2)

of (1.1) is that its solutions are not structurally stable. Suppose for example $u(0)$ and

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$v(0)$ are such that u and v for $\tau > 0$ are on the trajectory N_4 which passes close to the u and v axes. Then any small perturbation will move the solution onto another trajectory which does not lie everywhere close to the original one N_4. Thus a small perturbation can have a very marked effect, at the very least on the amplitude of the oscillation. In other words, this high sensitivity to the smallest change in the model leads to a model with essentially different species behaviour. For instance the positive equilibrium point of the system which is a center changes, under small pertubations, to a focus. For a 3-species Lotka-Volterra-type model without delays, as show in [5], there exist Hopf-bifurcating periodic orbits, but no conclusion about their orbital stability can be drawn by using the available procedures given in [10, 13] and others.

Thus, instead of relying on the instantanous density of one species (prey, say), because young preys do not participate in the reproductive process prior to sexual maturity, together with the fact that the predators are not usually successful in hunting young preys which do not wander away from their safe havens, we can get a more realistic model if we take into account density of one species (prey say) in the past, that is a system with delays.

There are several examples of 3-species communities such as linear chain of 3-species, see Fig. (1.3) (i); two preys and one predator, see Fig. (1.3) (ii) and two predators interacting with one prey as in Fig. (1.3) (iii).

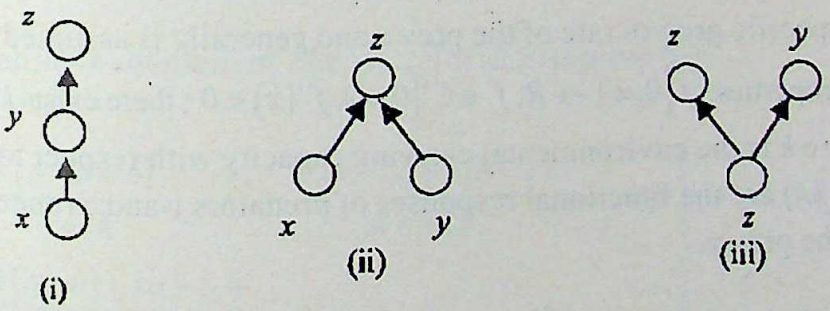

Fig. (1.3)

Since the question of two predatory populations consuming a single prey population has occupied an important place in ecological literature, thus in this paper we have a differential system modeling the interaction of two predators feeding upon a common prey. We have incorporated discrete delays $\tau \geq 0$ in the logistic maturation

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term of the prey species and $\sigma > 0$ in the predation terms and biomass conversion terms. Both predatury populations have been considered with density-dependent death rates. By using Lyapunov function technique as discussed for 2-dim predator-prey system in [9], we have obtained an estimation of delay's length for which stability is preserved, we have shown the existence of a critical value $\hat{\sigma}$ of σ (with $\tau = 0$) at which stability of the unique positive equilibrium switches to instability and the system remains unstable for $\sigma > \hat{\sigma}$.

MODEL WITHOUT DELAYS AND LOCAL ASYMPTOTIC STABILITY

An intermediate type 3-species system, see [7], modeling the inter-species interaction of two predators and one prey is of the form:

$$\left.\begin{aligned} \dot{x} &= xf(x) - yq_1(x) - zq_2(x) \\ \dot{y} &= y[-g(y) + cq_1(x)] \\ \dot{z} &= z[-h(z) + dq_2(x)] \end{aligned}\right\} \tag{2.1}$$

with

$$x(0) = x_0 > 0 \quad , \quad y(0) = y_0 > 0 \quad , \quad z(0) = z_0 > 0 \quad , \quad \cdot = \frac{d}{dt}$$

$f(x)$ is the specific growth rate of the prey x and generally is assumed to have the following properties: $f[0,\infty) \to R; f \in C^1[0,\infty), f'(x) < 0$; there exists $k>0$ such that $f(k)=0$ where k is the environmental carrying capacity with respect to the prey x. $q_1(x)$ and $q_2(x)$ are the functional responses of predators y and z repectively with respect to the prey x.

Generally, it is assumed that $q_1; [0,\infty) \to \to [0,\infty); q_i \in C^1[0,\infty); q_i(0) = 0, q_i'(x) > 0$ $i=1,2$.

$g(y)$ and $h(z)$ are the density-dependent specific death rates of predators y and z respectively. It is assumed that $g(0)>0$, $h(0)>0$, $g'(y) \geq 0$ for $y > 0$ and $h'(z) \geq 0$ for $z > 0$.

In this paper, however, we consider the folliwng special case of (2.1):

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$$\left.\begin{aligned} \dot{x} &= x\left[\in\left(1-\frac{x}{k}\right)-ay-bz\right] \\ \dot{y} &= y\left[-(1-y)+\frac{1}{a}x\right] \\ \dot{z} &= z\left[-(1+z)+\frac{1}{b}x\right] \end{aligned}\right\} \tag{2.2}$$

where the parameter $k>0$ is the above mentioned carrying capacity of the environment with respect to the prey x which, in turn, has a logistic growth rate, (i.e. it does not grow unboundedly in the absence of predators), $\in > 0$ is the intrinsic growth rate of the prey x in the absence of predation, $a>0$, $b>0$ are predation rates (i.e. functional responses of predators y and z respectively with respect to x), $\frac{1}{a}, \frac{1}{b}$ are biomass conversion rates. We, further; make the following assumptions:

$$\left.\begin{aligned} & 0<a<b<1 \quad , \quad k, \in > b > a, \\ \text{and} \quad & ka - \in b > 0 \quad if \quad \in \neq k \\ \text{and} \quad & k + a > 2b \quad if \quad \in = k. \end{aligned}\right\} \tag{2.3}$$

The unique positive equilibrium $E(x^*, y^*, z^*)$ of (2.2) is given by:

$$\left.\begin{aligned} x^* &= \frac{k(\in + a + b)}{\in + 2k}, \quad y^* = \frac{\in(k-a)+k(b-a)}{a(\in+2k)}, \quad and \\ z^* &= \frac{k(\in - b)+ka-b\in}{b(\in+2k)} \end{aligned}\right\} \tag{2.4}$$

It is straightforward to show that the eigenvalues of the variational matrix $V(E)$ of the system (2.2) about $E(x^*, y^*, z^*)$ are given by the characteristic equation:

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + P_3 = 0 \tag{2.5}$$

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where

$$\left.\begin{aligned} p_1 &= x^*\frac{\in}{k} + y^* + z^* \\ p_2 &= x^*y^*\left(1+\frac{\in}{k}\right) + x^*z^*\left(1+\frac{\in}{k}\right) + y^*z^* \\ p_3 &= x^*y^*z^*\left(2+\frac{\in}{k}\right). \end{aligned}\right\} \tag{2.6}$$

Since $P_1>0$, $P_3>0$, and

$$p_1p_2 - p_3 = \frac{\in}{k}\left(1+\frac{\in}{k}\right)\left(x^*\right)^2\left(y^*+z^*\right) + x^*\left(1+\frac{\in}{k}\right)\left[\left(y^*\right)^2+\left(z^*\right)^2\right]$$
$$+y^*z^*\left(y^*+z^*\right)+\frac{2\in}{k}x^*y^*z^* > 0,$$

then by Routh-Hurwitz criterion $E\left(x^*,y^*,z^*\right)$ given by (2.4) is locally asymptotically stable.

MODEL WITH DELAYS AND LOCAL ASYMPTOTIC STABILITY

We incorporate discrete delays in the system (2.2) as follows:

$$\left.\begin{aligned} \dot{x} &= x(t)\left[\in\left(1-\frac{x(t-\tau)}{k}\right)\right] - ay(t-\sigma) - bz(t-\sigma) \\ \dot{y} &= y(t)\left[-(1+y(t)) + \frac{1}{a}x(t-\sigma)\right] \\ \dot{z} &= z(t)\left[-(1-z(t)) + \frac{1}{b}x(t-\sigma)\right] \end{aligned}\right\} \tag{3.1}$$

with the initial conditions:

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$$\left.\begin{aligned} x(t) &= \varnothing_1(t), t \in [-r,0], \varnothing_1 \in C([-r,0], R^+) \\ y(t) &= \varnothing_2(t), t \in [-r,0], \varnothing_2(t) \in C([-r,0], R^+) \\ z(t) &= \varnothing_3(t), t \in [-r,0], \varnothing_3(t) \in C([-r,0], R^+) \end{aligned}\right\} \tag{3.2}$$

where $r = \max\{\tau, \sigma\}$.

Clearly the system (3.1) with (3.2) has a unique solution which exits and is positive for all $t \geq 0$. τ denotes delay incorporated in the logistic growth term of the prey species. To make calculation simpler we have taken the delay in both predation terms in the prey growth rate equation and in the biomass conversion terms in both predators growth rate equations to be the same ($\sigma > 0$ say), otherwise the analysis technique remains the same with different delays.

Naturally $E(x^*, y^* z^*)$ as given by (2.4) remains the unique positive equilibrium of the system (3.1). Let us introduce a new coordinate system through the following substitution:

$$x_1(t) = x(t) - x^*, \qquad y_1(t) = y(t) - y^* \qquad z_1(t) = z(t) - z^*.$$

From (3.1), $(x_1(t), y_1(t), z_1(t))$ satisfies the system:

$$\left.\begin{aligned} \dot{x}_1(t) &= (x_1(t) + x^*)x^*\left[-\frac{\in}{k} x_1(t-\tau) - a y_1(t-\sigma) - b z_1(t-\sigma)\right] \\ \dot{y}_1(t) &= (y_1(t) + y^*)y^*\left[-y_1(t) + \frac{1}{a} x_1(t-\sigma)\right] \\ \dot{z}_1(t) &= (z_1(t) + z^*)z^*\left(-z_1(t) + \frac{1}{b} x_1(t-\sigma)\right) \end{aligned}\right\} \tag{3.3}$$

Remark 3.1.

The stability analysis of $E(x^*, y^* z^*)$ is reduced to that of zero solution of (3.3). From (3.3), it follows that the variational system of (3.1) with respect to the positive

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equilibrium $E(x^*, y^*, z^*)$ is given by the linearized system of (3.3) as:

$$\left.\begin{aligned}
\dot{x}_1(t) &= x^*\left[-\frac{\in}{k}x_1(t-\tau) - ay_1(t-\sigma) - bz_1(t-\sigma)\right] \\
\dot{y}_1(t) &= y^*\left[-y_1(t) + \frac{1}{a}x_1(t-\sigma)\right] \\
\dot{z}_1(t) &= z^*\left(-z_1(t) + \frac{1}{b}x_1(t-\sigma)\right)
\end{aligned}\right\} \tag{3.4}$$

Now, we can rewrite (3.4) as follows:

$$\left.\begin{aligned}
&\frac{d}{dt}\left[\frac{x_1(t)}{x^*} - \frac{\in}{k}\int_{t-\tau}^{t} x_1(s)ds - a\int_{t-\sigma}^{t} y_1 ds - b\int_{t-\sigma}^{t} z_1(s)ds\right] = -\frac{\in}{k}x_1(t) - ay_1(t)\underline{b}z_1(t) && (i)\\
&\frac{d}{dt}\left[\frac{y_1(t)}{y^*} + \frac{1}{a}\int_{t-\sigma}^{t} x_1(s)ds\right] = y_1(t) + \frac{1}{a}x_1(t) && (ii)\\
&\frac{d}{dt}\left[\frac{z_1(t)}{z^*} + \frac{1}{b}\int_{t-\sigma}^{t} x_1(s)ds\right] = \underline{z}_1(t) + \frac{1}{b}x_1(t) && (iii)
\end{aligned}\right\} \tag{3.5}$$

Now, we construct a Lyapunov function $w(t)$ so that the trivial solution (i.e. (0,0,0)) of (3.3) is stable and then we show solution $(x_1(t), y_1(t), z_1(t)) \to (0, 0, 0)$ as $t \to \infty$ which implies that (0,0,0) is a locally asymptotically stable solution of (3.3), which consequently implies that $E(x^*, y^*, z^*)$ is locally asymptotically stable.

Consider
$$w(t) = x^* w_1(t) + y^* a^2 w_2(t) + z^* b^2 w_3(t), \tag{3.6}$$

Where
$$w_1(t) = w_{11}(t) + w_{12}(t) \tag{3.7}$$

For
$$w_{11}(t) = \left[\frac{x_1(t)}{x^*} - \frac{\in}{k}\int_{t-\tau}^{t} x_1(s)ds - a\int_{t-\sigma}^{t} y_1(s)ds - b\int_{t-\sigma}^{t} z_1(s)ds\right]^2 \tag{3.8}$$

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$$w_{12}(t)=\left(\frac{\in}{k}+a+b\right)\left[\frac{\in}{k}\int_{t-\tau}^{t}\int_{s}^{t}x_1^2(l)dlds+a\int_{t-\sigma}^{t}\int_{s}^{t}y_1^2(l)dlds+b\int_{t-\sigma}^{t}\int_{s}^{t}z_1^2(l)dlds\right] \quad (3.9)$$

where $w_2(t)=w_{21}(t)+w_{22}(t)$ (3.10)

for $$w_{12}(t)=\left[\frac{y_1(t)}{y^*}+\frac{1}{a}\int_{t-\sigma}^{t}x_1(s)ds\right] \quad (3.11)$$

for $$w_{22}(t)=\left(\frac{1}{2}+\frac{1}{a^2}\right)\int_{t-\sigma}^{t}\int_{s}^{t}x_1^2(l)dlds \quad (3.12)$$

and where $$w_3(t)=w_{31}(t)+w_{32}(t) \quad (3.13)$$

for $$w_{31}(t)=\left[\frac{z_1(t)}{z^*}+\frac{1}{b}\int_{t-\sigma}^{t}x_1(s)ds\right]^2 \quad (3.14)$$

$$w_{32}(t)=\left(\frac{1}{b}+\frac{1}{b^2}\right)\int_{t-\sigma}^{t}\int_{s}^{t}x_1^2(l)dlds \quad (3.15)$$

Remarks 3.2.

Let $X(t)=\left(x_1(t),y_1(t),z_1(t)\right)$. Thus by $w(t)$ wen mean: $w(t)=w(X)(t)$, and similarly for $w_1(t),w_2(t),w_3(t),w_{11}(t),w_{12}(t),w_{21}(t),w_{22}(t),w_{31}(t)$ and $w_{32}(t)$.

From the defination of $w(t)$, it is clear that $w(t)$ is positive definite.

Next we compute the time derivative of $w(t)$ with respect to (3.4). As shown in the Appendix we get:

$$\frac{dw(t)}{dt}\le x_1^2(t)\frac{\in}{k}\left[-2+x^*\left(\frac{2\in}{k}\tau+(a+b)(\tau+\sigma)\right)+\frac{k}{\in}y^*\sigma(a+2)+\frac{k}{\in}\sigma(b+2)z^*\right]$$

$$+a^2y_1^2(t)\left[-2+\frac{x^*}{a}\left(\frac{\in}{k}(\tau+\sigma)+2\sigma(a+b)\right)+\frac{\sigma}{a}y^*\right]$$

$$+b^2z_1^2(t)\left[-2+\frac{x^*}{b}\left(\frac{\in}{k}(\tau+\sigma)+2\sigma(a+b)\right)+\frac{\sigma}{b}z^*\right]$$

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Thus

$$\frac{dw(t)}{dt} \le -\frac{\in}{k}\alpha_1 x_1^2(t) - a^2\alpha_2 y_1^2(t) - b^2\alpha_3 z_1^2(t), \tag{3.16}$$

where it is assumed that:

$$\left.\begin{aligned} a_1 &= 2 - \left\{ x^*\left(\frac{2\in}{k} + a + b\right)\tau + \left[x^*(a+b) + \frac{k}{\in} y^*(a+2) + \frac{k}{\in} z^*(b+2)\right]\sigma \right\} > 0 \quad (i) \\ a_2 &= 2 - \left\{ \left(\frac{x^*\in}{ak}\right)\tau + \frac{1}{a}\left[x^*\left(\frac{\in}{k} + 2a + 2b\right) + y^*\right]\sigma \right\} > 0 \quad (ii) \\ a_3 &= 2 - \left\{ \left(\frac{x^*\in}{bk}\right)\tau + \frac{1}{b}\left[x^*\left(\frac{\in}{k} + 2a + 2b\right) + z^*\right]\sigma \right\} > 0 \quad (iii) \end{aligned}\right\} \tag{3.17}$$

let $$a = \min\left\{ \frac{\in}{k} a_1 \alpha^2 \alpha_2, b^2\alpha_3 \right\} \tag{3.18}$$

Thus (3.16) gives:

$$\frac{dw(t)}{dt} \le -a\left(x_1^2(t) + y_1^2(t) + z_1^2(t)\right). \tag{3.19}$$

From (3.19) it follows that $w(x_1(t), y_1(t), z_1(t))$ as defined by (3.6) is the sought after Lyapunov function. To prove local asymptotic stability of $E(x^*, y^*, z^*)$ it remains to show that:

$$\lim_{t\to\infty}\left[x_1^2(t) + y_1^2(t) + z_1^2(t)\right] = 0.$$

Now integrating (3.19) from T to t, we get

$$w(t) + a\int_T^t \left[x_1^2(s) + y_1^2(s) + z_1^2(s)\right] ds \le w(T), \qquad t \ge T \tag{3.20}$$

But (3.20) implies that $x_1^2(t) + y^2(t) + z^2(t) \in L[T, \infty)$.

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We claim that $\lim_{t\to\infty}\left[x_1^2(t)+y_1^2(t)+z_1^2(t)\right]=0$.

Since $w(t)$ is a decreasing function with strict minimum at (0,0,0), $\lim_{t\to\infty} w(t)$ exists and it finite.

This is because $\lim_{t\to\infty}\left[x_1^2(t)+y_1^2(t)+z_1^2(t)\right]=N\neq 0$, will according to (3.20) lead to a contradiction. Thus it follows that the (0,0,0) solution of (3.3) is locally asymtotically stable provided the following three conditions hold:

$$\left.\begin{aligned} &x^*\left(\frac{2\in}{k}+a+b\right)\tau+\left[x^*(a+b)+y^*\frac{k}{\in}(a+2)+z^*\frac{k}{\in}(b+2)\right]\sigma<2 && (i)\\ &\left(\frac{x^*\in}{ak}\right)\tau+\left[\frac{x^*}{a}\left(\frac{\in}{k}+2a+2b\right)+\frac{y^*}{a}\right]\sigma<2 && (ii)\\ &\left(\frac{x^*\in}{bk}\right)\tau+\left[\frac{x^*}{b}\left(\frac{\in}{k}+2a+2b\right)+\frac{z^*}{b}\right]\sigma<2 && (iii)\end{aligned}\right\} \quad (3.21)$$

Now, we determine values of τ and σ satisfying the three inequalities (3.2.1) (i), (ii) and (iii).

Since from (2.3) we have $\frac{1}{b}<\frac{1}{a}$, and fromn (2.4) it follows $y^*>z^*$ and thus $\frac{y^*}{a}>\frac{z^*}{b}$, hence it is sufficient to find values τ and σ satisfying (3.2.1) (i) and (ii) only. Thus we have proved the following main theorem:

Theorem (3.1)

Assume that (2.3) holds. If the delays τ and σ satisfy the inequalities (3.2.1) (i) and (3.2.1) (ii), then the positive equilibrium $E(x^*,y^*,z^*)$ is locally asymtotically stable.

Let us consider the following illustrative example:

Example 3.1.

If we let $\in=k=1, a=\frac{5}{8}, b=\frac{6}{8}$, then from (2.4) we have $x^*=\frac{19}{24}, y^*=\frac{4}{15}$,

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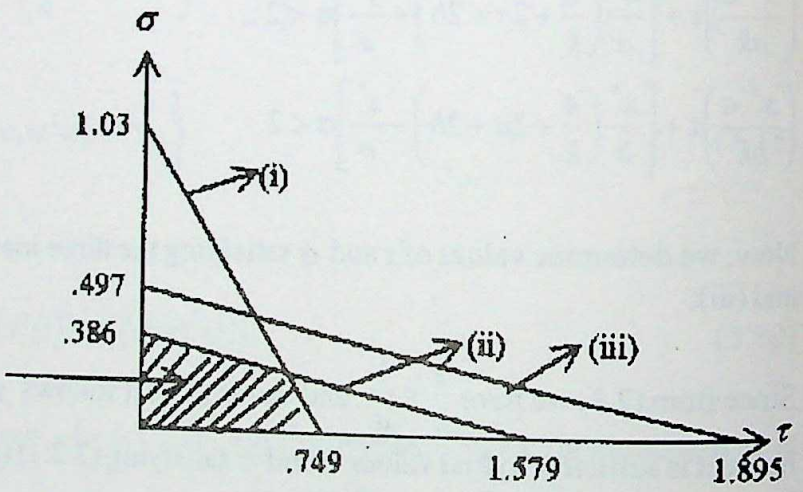
and $z^* = \frac{1}{8}$, consequently the inequalities (3.2.1) (i), (ii) and (iii) become:

$$\left.\begin{aligned} (2.67)\tau + (1.94)\sigma < 2 \quad & (i) \\ (1.266)\tau + (5.176)\sigma < 2 \quad & (ii) \\ (1.055)\tau + (4.022)\sigma < 2 \quad & (iii) \end{aligned}\right\} \tag{3.22}$$

Thus as long as

$$\left.\begin{aligned} 0 \le \tau < .749 \quad \text{and} \\ 0 \le \sigma < .386, \end{aligned}\right\} \tag{3.23}$$

the equilibrium $E\left(\frac{19}{24}, \frac{4}{15}, \frac{1}{18}\right)$ is locally asymtotically stable, see Fig. 3.1.



Remark 3.3.

It follows from (3.23) that the delay $\sigma > 0$ in the predation and biomass conversion terms is more destabilizing than the maturation delay $\tau \ge 0$ of the prey as far as this particular Lotka-Volterra predator-prey model is concerned.

STABILITY SWITCHING

In this section, we show the existence of some critical value $\hat{\sigma} > 0$ of the delay $\sigma > 0$ (with $\tau = 0$) such that for $\sigma > \hat{\sigma}$ there occurs stability switching of the equilibrium $E(x^*, y^*, z^*)$ from stability to instability, in other words the equilibrium $E(x^*, y^*, z^*)$ is

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unstable for $\sigma > \hat{\sigma}$. The discussion is based upon Theorem 1 of [2] and corollary 4.3. of [6].

We consider the system (3.1) with $\tau = 0$ and $\in = k = 1$, that is

$$\left.\begin{aligned} \dot{x} &= x(t)\left[(1-x(t)) - ay(t-\sigma) - bz(t-\sigma)\right] \\ \dot{y} &= y(t)\left[-(1+y(t)) + \frac{1}{a}x(t-\sigma)\right] \\ \dot{z} &= z(t)\left[-(1+z(t)) + \frac{1}{b}x(t-\sigma)\right] \end{aligned}\right\} \tag{4.1}$$

The characteristic equation for the linearization of (4.1) is

$$\lambda^3 + a\lambda^2 + \beta\lambda + \gamma = (\delta + \lambda\theta)e^{-2\lambda\sigma} \tag{4.2}$$

where,

$$\left.\begin{aligned} \alpha &= x^* + y^* + z^* \\ \beta &= x^*(y^* + z^*) + y^* z^* \\ \gamma &= x^* y^* z^* \\ \delta &= -2x^* y^* z^* \\ \theta &= -x^*(y^* + z^*) \end{aligned}\right\} \tag{4.3}$$

Let $P(\lambda) = \lambda^3 + a\lambda^2 + \beta\lambda + \gamma$

$Q(\lambda) = -(\lambda\theta + \delta)$

Since P and Q are polynomials with real coefficients, the condition (HI) of Theorem 1 of [2] is satisfied.

Now

$$P(\lambda) + Q(\lambda) \equiv \lambda^3 + a\lambda^2 + \beta\lambda + \gamma - \lambda\theta - \delta = 0 \quad \text{or}$$

$$\lambda^3 + \left[x^* + y^* + z^*\right]\lambda^2 + \left[2x^*(y^* + z^*) + y^* z^*\right] + 3x^* y^* z^* = 0 \tag{4.4}$$

Now (4.4) is same as (2.5) with (2.6).

Thus as shown in section 2, $E(x^*, y^*, z^*)$ is locally asymptotically stable. This implies

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that $P(\lambda)+Q(\lambda)=0$ has no root in $\mathrm{Re}(\lambda)\geq 0$. Thus condition (H2) of Theorem 1 of [2] is satisfied. In order to utilize Theorem 1 of [2], let us define:

$$\begin{aligned} F(y) &= |P(iy)|^2 - |Q(iy)|^2 \\ &= y^6 + (\alpha^2 - 2\beta)y^4 + (\beta^2 - 2a\gamma - \theta^2)y^2 + (\gamma^2 - \delta^2) \end{aligned} \tag{4.5}$$

Where

$$\left.\begin{aligned} a^2 - 2\beta &= (x^*)^2 + (y^*)^2 + (z^*)^2 > 0 \\ F(0) &= \gamma^2 - \delta^2 = -3(x^* y^* z^*)^2 < 0 \\ \beta^2 - 2a\gamma - \theta^2 &= y^* z^* (y^* z^* - 2(x^*)^2) \end{aligned}\right\} \tag{4.6}$$

Now Theorem 1 of [2] is stated as follows:
Assume conditions ($H1$) and ($H2$) hold.

(a) If $F(y)$ has no positive roots, then no stability switches occur. (This implies either the equilibrium point is stable for all $\sigma \geq 0$ or it is unstable for all $\sigma \geq 0$.

(b) If $F(y)$ has positive roots, stability switches may occur. At most a finite number of switches is possible. There exists $\hat{\sigma} \geq 0$ or such that the equilibrium point is unstable for all $\sigma \geq \hat{\sigma}$.

Now since $a^2 - 2\beta > 0$ and $\gamma^2 - \delta^2 < 0$, then by Descartes rule of sign applied to $F(y)$ as given by (4.5), the possibility that $F(y)>0$ for all $y>0$ (i.e. $F(y)$ has no positive root) is completely ruled out. This implies that the stability of the equilibrium point $E(x^*, y^*, z^*)$ is not preserved for all $\sigma > 0$.

Thus we have following theorem:

Theorem 4.1.

Assume (2.3) (i.e. the equilibrium $E(x^*, y^*, z^*)$ of (4.1) with $\sigma = 0$ is asymptotically stable). Then there exists $\hat{\sigma} > 0$ for which $E(x^*, y^*, z^*)$ is unstable for $\sigma > \hat{\sigma}$.

Remark 4.1.

Prior ultimately shifting to instability for $\sigma > \hat{\sigma}$, there may occur at most a finite

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number of stability switches.

Example 4.1.

Consider the system (4.1). The associated linearized system has the characteristic equation:

$$\lambda^3 + a\lambda^2 + \beta\lambda + \gamma - (\theta\lambda + \delta)e^{-2\lambda\sigma} = 0.$$

Let $a = \frac{5}{8}, b = \frac{6}{8}, \in = k = 1,$ we have $x^* = \frac{19}{24}, y^* = \frac{4}{15}, z^* = \frac{1}{18}$. Thus from (4.3) we get

$$a = 1.114,\ \beta = 0.2699,\ \gamma = 0.0117,\ \delta = -0.023457,\ \theta = -0.25499, \tag{4.7}$$

From (4.5), we have:

$$F(y) = y^6 + 0.70119y^4 - 0.018234y^2 - 0.0004107,$$

which has positive real root y_1=0.19724.

This indicates that preservation of stability can not hold for all $\sigma > 0$.

Thus there exists $\hat{\sigma}$ such that for all $\sigma > \hat{0}$, the equilibrium is unstable.

Remarks 4.2.

The delay which could represent in our model influence of the past or waiting period in the proliferation of certain species due to (say) lag in some physiological process (because of natural or unnatural reasons), could have if longer than enough a destabilizing effect on the otherwise stable equilibrium coexistence of the ecosystem's species.

Remark 4.3.

In case the two delays τ and σ are kept in the model (4.1) as in case of (3.1), then the corresponding characteristic equation is:

$$\lambda^3 + (y^* + z^*)\lambda^2 + y^*z^*\lambda + \left[\lambda^2 x^* + \lambda x^*(\lambda^* + z^*) + x^*y^*z^*\right]e^{-\lambda\tau}$$
$$+\left[\lambda x^*(y^* + z^*) + 2x^*y^*z^*\right]e^{-2\lambda\sigma} = 0.$$

However, discussion of the full analysis of stability switches for arbitary σ and τ remains an open problem, see [2,3].

APPENDIX

In this apprendix we prove (3.16). From (3.8), the time derivative of $w_{11}(t)$ with

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respect to (3.4) while using (3.5) is:

$$\frac{dw_{11}(t)}{-dt} = 2\left[\frac{x_1(t)}{x^*}\frac{-\in}{k}\int_{t-\tau}^{t} x_1(s)ds - a\int_{t-\sigma}^{t} y_1(s)ds - b\int_{t-\sigma}^{t} z_1(s)ds\right]$$

$$\times\left[\frac{-\in}{k}x_1(t) - ay_1(t) - bz_1(t)\right]$$

$$= \frac{-2\in}{k}\frac{x_1^2(t)}{x^*} - \frac{2a}{x^*}x_1(t)y_1(t) - \frac{2b}{x^*}x_1(t)z_1(t)$$

$$+2\left(\frac{\in}{k}\right)^2\int_{t-\tau}^{t} x_1(t)x_1(s)ds + \frac{2\in a}{k}\int_{t-\sigma}^{t} x_1(t)y_1(s)ds + \frac{2\in b}{k}\int_{t-\sigma}^{t} x_1(t)z_1(s)ds$$

$$+\frac{2\in a}{k}\int_{t-\tau}^{t} y_1(t)x_1(s)ds + 2a^2\int_{t-\sigma}^{t} y_1(t)y_1(s)ds + 2ab\int_{t-\sigma}^{t} y_1(t)z_1(s)ds$$

$$+\frac{2\in b}{k}\int_{t-\tau}^{t} z_1(t)x_1(s)ds + 2ab\int_{t-\sigma}^{t} z_1(t)y_1(s)ds + 2b^2\int_{t-\sigma}^{t} z_1(t)z_1(s)ds$$

$$\le \frac{-2\in}{k}\frac{x_1^2(t)}{x^*} - \frac{2a}{x^*}x_1(t)y_1(t) - \frac{2b}{x^*}x_1(t)z_1(t)$$

$$+\left\{\left(\frac{\in^{8^2}}{k}\right)\left[x_1^2(t)\tau + \int_{t-r}^{t} x_1^2(s)ds\right] + \frac{\in}{k}a\left[x_1^2(t)\sigma + \int_{t-\sigma}^{t} y_1^2(s)ds\right]\right.$$

$$+\frac{\in b}{k}\left[x_1^2(t)\sigma + \int_{t-\sigma}^{t} z_1^2(s)ds\right] + \frac{\in a}{k}\left[y_1^2(t)\tau + \int_{t-\tau}^{t} x_1^2(s)ds\right]$$

$$+a^2\left[y_1^2(t)\sigma + \int_{t-\sigma}^{t} y_1^2(s)ds\right] + ab\left[y_1^2(t)\sigma + \int_{t-\sigma}^{t} z_1^2(s)ds\right]$$

$$\left.+\frac{\in b}{k}\left[z_1^2(t)\tau + \int_{t-r}^{t} x_1^2(s)ds\right] + ab\left[z_1^2(t)\sigma + \int_{t-\sigma}^{t} y_1^2(s)ds\right] + b_2\left[z_1^2(t)\sigma + \int_{t-\sigma}^{t} z_1^2(s)ds\right]\right\} \quad \text{(A-1)}$$

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Define $w_1(t) = w_1(X)(t) = w_{11}(X)(t) = w_{12}(X)(t),$ (A-2)

Where $w_{12}(t)$ is defined so that the integral terms in (A-1) get eliminated. Thus

$$w_{12}(t) = \left(\frac{\in}{k} + a + b\right)\left[\frac{\in}{k}\int_{t-\tau}^{t}\int_{s}^{t} x_1^2(l)dlds + a\int_{t-\sigma}^{t}\int_{s}^{t} + b\int_{t-\sigma}^{t}\int_{s}^{t} z_1^2(l)dlds\right] \quad \text{(A-3)}$$

Using the fact that $\int_{t-r}^{t} x^2(s)ds + \frac{d}{dt}\left[\int_{t-r}^{t}\int_{s}^{t} x^2(l)dlds\right] = x^2(t)r$, then from (A-1), (A-2) and (A-3), we get:

$$\begin{aligned}\frac{dw_1(t)}{dt} \le{}& \frac{-2 \in}{k}\frac{x_1^2(t)}{x^*} - \frac{2a}{x^*}x_1(t)y_1(t) - \frac{2b}{x^*}x_1(t)z_1(t)\\ &+\left(\frac{\in}{k}\right)^2\left[x_1^2(t)\tau + x_1^2(t)\tau\right] + \frac{\in a}{k}\left[x_1^2(t)\sigma + y_1^2(t)\sigma\right]\\ &+\frac{\in b}{k}\left[x_1^2(t)\sigma + z_1^2(t)\sigma\right] + \frac{\in a}{k}\left[y_1^2(t)\sigma + x_1^2(t)\tau\right]\\ &+a^2\left[y_1^2(t)\sigma + y_1^2(t)\sigma\right] + ab\left[y_1^2(t)\sigma + z_1^2(t)\sigma\right]\\ &+\frac{\in b}{k}\left[z_1^2(t)\tau + x_1^2(t)\tau\right] + ab\left[z_1^2(t)\sigma + y_1^2(t)\sigma\right]\\ &+b^2\left[z_1^2(t)\sigma + z_1^2(t)\sigma\right]\end{aligned}$$

Thus, we have:

$$\begin{aligned}\frac{dw_1(t)}{dt} \le{}& x_1^2(t)\frac{\in}{k}\left[\frac{-2}{x^*} + \frac{2 \in \tau}{k} + (1+b)(\tau+\sigma)\right]\\ &+y_1^2(t)a\left[\frac{\in}{k}(\tau+\sigma) + 2\sigma(a+b)\right]\\ &+z_1^2(t)b\left[\frac{\in}{k}(\tau+\sigma) + 2\sigma(a+b)\right]\end{aligned}$$

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$$-\frac{2a}{x^*}x_1(t)y_1(t)-\frac{2b}{x^*}x_1(t)z_1(t). \tag{A-4}$$

Next, we define w_{21} as in (3.11):

$$w_{21}(t)=\left[\frac{y_1(t)}{y^*}+\frac{1}{a}\int_{t-\sigma}^{t}x_1(s)ds\right]^2.$$

From (3.11) and (3.5) (ii) we have

$$\frac{dw_{21}(t)}{dt}=21\left[\frac{y_1(t)}{y^*}+\frac{1}{a}\int_{t-\sigma}^{t}x_1(s)ds\right]\left[-y_1(t)+\frac{1}{a}x_1(t)\right]$$

$$=\frac{-2y_1^2(t)}{y^*}+\frac{2}{a}\frac{1}{y^*}x_1(t)y_1(t)-\frac{2}{a}\int_{t-\sigma}^{t}y_1(t)x_1(s)ds+\frac{2}{a^2}\int_{t-\sigma}^{t}x_1(t)x_1(s)ds$$

$$\le\frac{-2y_1^2(t)}{y^*}+\frac{2}{a}\frac{1}{y^*}x_1(t)y_1(t)+\frac{1}{a}\left[y_1^2(t)\sigma+\int_{t-\sigma}^{t}x_1^2(s)ds\right]$$

$$+\frac{1}{a^2}\left[x_1^2(t)\sigma+\int_{t-\sigma}^{t}x_1^2(s)ds\right]. \tag{A-5}$$

Define $$w_2(t)=w_2(X)(t)=w_{21}(X)(t)+w_{22}(X)(t) \tag{A-6}$$

Where $w_{22}(t)$ is defined so that the integral terms in (A-5) get eliminated. Thus

$$w_{22}(t)=\left(\frac{1}{a}+\frac{1}{a^2}\right)\int_{t-\sigma}^{t}\int_{s}^{t}x_1^2(l)dlds \tag{A-7}$$

From (A-5), (A-6) and (A-7), we get

$$\frac{dw_2(t)}{dt}\le x_1^2(t)\sigma\left[\frac{1}{a}+\frac{2}{a^2}\right]+y^2(t)\left[-\frac{2}{y^*}+\frac{\sigma}{a}\right]+\frac{2}{a}\frac{1}{y^*}x_1(t)y_1(t). \tag{A-8}$$

Similarly, we have:

$$w_3(t)=w_3(X)(t)=w_{31}(X)(t)+w_{32}(X)(t)$$

Where

$$w_{31}(t)=\left[\frac{z_1(t)}{z^*}+\frac{1}{b}\int_{t-\sigma}^{t}x_1(s)ds\right]^2$$

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and $w_{32}(t) = \left(\frac{1}{b} + \frac{1}{b^2}\right) \int_{t-\sigma}^{t} \int_{s}^{t} x_1^2(l) dl ds,$

and we get

$$\frac{dw_3(t)}{dt} \le x_1^2(t)\sigma\left[\frac{1}{b} + \frac{2}{b^2}\right] + z_1^2(t)\left[\frac{-2}{z^*} + \frac{\sigma}{b}\right] + \frac{2}{b}\frac{1}{z^*} x_1(t) z_1(t) \qquad \text{(A-9)}$$

From (3.6) and using (A-4), (A-8) and (A-9) we get (3.16).

This completes the proof.

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A TWO WAREHOUSE DETERMINISTIC ECONOMIC ORDER QUANTITY MODEL WITH FINITE RATE OF REPLENISHMENT AND STOCK DEPENDENT

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ABSTRACT

A deterministic inventory model is developed for a single item with two warehouses one owned by the management (abbreviated as OW) and other is a rented warehouse (abbreviated as RW) by considering stock dependent consumption rate. It is assumed that storage capacity of OW is limited and that of RW is unlimited. The holding cost of RW is greater than OW and actual service of the customers is being done from OW, so it is economical to empty RW first. It is assumed that stocks are transferred from RW to OW according to K-Release Rule (KRR) in such a way that units in each transshipment is less than the capacity of OW. The model is set up by assuming finite replenishment rate and shortages are not allowed. Finally the solution of the model is discussed with the help of a numerical example.

Key words & Phrases : Demand, EOQ, Inventory, finite replenishment, L_2-system.
AMS Classification No. : 90-XX.

INTRODUCTION

Most of the inventory models are developed for a single warehouse. Due to large stock and limited capacity of existing storage facility *OW*, an additional storage facility *RW* which is located away from *OW* is hired to store excess items. In practice large stock attracts the management to either an attractive price discount for bulk purchase or the management resorts this practice where the acquisition costs are higher than the holding cost in *RW* or due to existing storage capacity limitations.

One warehouse, inventory storage facility is known as L_1- system whereas a two warehouse inventory storage facility is known as L_2- system. A good number of authors have considered L_2- system under different assumptions. Hartley [5]

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discussed this type of model neglecting the cost of transporting a unit from RW to OW. Sarma [11] extended it by considering the transportation cost of a unit from RW to OW with K-release rule and improved the working of the model [12, 13]. Dave [2] also improved the working of the model given by Sarma [11]. Murdeshwar and Sathe [7] extended the model given by Sarma [11] by considering finite production rate. Gupta and Vrat [4] developed a single level inventory model with stock dependent consumption rate. Mendus and Antony [8] developed a two warehouse inventory model by considering stock dependent consumption rate and infinite replenishment rate.

In this paper a L_2- system with stock dependent consumption rate with finite rate of replenishment with K- release rule (KRR) is considered, as large stock attracts customers particularly in consumer goods. Realistic assumptions given in this paper make the model more practical.

ASSUMPTIONS AND NOTATIONS

The following assumptions and notations are used for the development of the model.

(i) D is the demand rate and is taken as $D = \alpha + \lambda Q^{\delta} (\alpha, \lambda, \delta \text{ are constant})$.

(ii) Q is the highest inventory level.

(iii) The items of RW are transferred to OW in 'n' shipment of which $K \leq W$ units are transported in each shipment.

(iv) Replenishment/Production rate is finite. Let 'R' be the finite replenishment rate per time unit, which is known, constant and uniform during the period under consideration.

(v) Storage capacity of OW is W and that of RW is unlimited.

(vi) Let $\beta = 1 - D/R$ and $W_1 = W/\beta$.

(vii) Lead time is zero.

(viii) Set up cost is fixed and taken as A.

(ix) The transportation cost of K units from RW to OW is C_t at a time, which is constant over time.

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(x) The holding cost/unit in OW is H and that in RW is F and $F>H$.

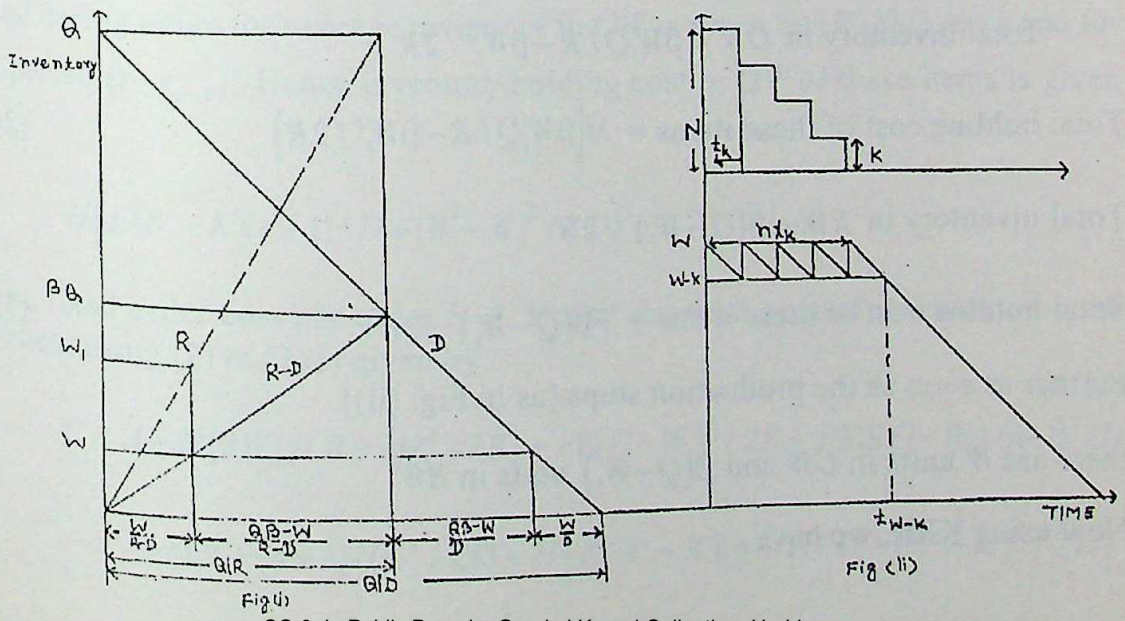
(xi) Shortages are not allowed.

(xii) Consumption takes place from OW only.

(xiii) T represents the total time period (cycle length).

MATHEMATICAL MODEL AND ANALYSIS FOR THE SYSTEM

We assume that a company purchases/products Q units, the finite rate of replenishment/production is R. During production stage the demand is satisfied from production and the excess quantities are transferred to warehouses as in Fig. (i). It is economical to fill OW first and when it is filled to its maximum capacity then excess production/replenishment is transferred to RW. When the production stops, the accumulated inventory can be used to satisfy further demand from OW. Due to this, inventory level in OW will fall from its maximum capacity W. When the stock level in OW reaches to $(W\text{-}K)$ units, at this stage 'K' units from RW are transported to OW to meet further demand and this process is repeated 'n' times untill stock in RW is fully exhausted. The remaining $(W\text{-}K)$ units in OW are used again at this stage. Also noting that after production stops, the inventory level in OW is 'W' and that in RW is $z=(Q\beta-W)$ units. The inventory level is shown in figure (ii).



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Case (i) (Before Consumption)

Initially, we have the total average cost as

$$C(Q) = \frac{AD}{Q} + \frac{HW}{2} + \frac{F(\beta Q - W)}{2} \qquad (1)$$

Where $D = \alpha + \lambda Q^{\delta}$

For optimum value Q^* of Q, which minimize total average cost is obtained by solving

$dC(Q)/dQ = O$ which gives $Q^* = \sqrt{\frac{2A\alpha}{\beta F}}$ $\quad (\text{If } \lambda = 0)$

As $\beta \to 1$ i.e. for infinite rate of replenishment

$Q^* = \sqrt{\frac{2A\alpha}{F}}$ which is same as Classical EOQ formula.

Case (ii) (During Consumption)

From figure (1). It is clear that during production stage.

Total inventory in $OW = \beta W_1 Q/R - \beta W_1^2/2R$

Total holding cost of these items = $H\left[\beta W_1 Q/R - \beta W_1^2/2R\right]$ (2)

Total inventory in $RW = \beta(Q - W_1)^2/2R$.

Total holding cost of these items = $F\beta(Q - W_1)^2/2R$. (3)

Further as soon as the production stops [as in Fig. (ii)]

there are W units in OW and $\beta(Q - W_1)$ units in RW.

Now using KRR, we have

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Inventory units in RW are given by

$$= t_k\left[Z+(Z-K)+(Z-2K)+.....+Z-(n-1)K\right]$$

$$= \frac{t_k(n+1)Z}{2}$$

where $t_k = K/D$, time taken for the consumption of K units.

Total holding cost of these items in $RW = F\beta k(Q-W_1)(n+1)/2D$ (4)

Cost of transporting K units from

RW to OW is given by $= nC_t = \beta(Q-W_1)C_t/K$ (5)

When K units are drawn from RW in each shipment more are carried in OW for a period of t_k and hence holding cost of these items is given by $KHt_k/2$.

Since there are 'n' such shipment and taking the initial K units into account, we have The total holding cost of these items

$$= \left\{(n+1)HKt_k\right\}/2 = \left\{(n+1)HK^2\right\}/2D \quad (6)$$

A quantity of (W-K) units is kept unused in OW for a period of $t_{w-k} = (n+1)t_k$ and during usage the average inventory in OW is given by (W-K)/2 units and for a period $(t-t_{w-k})$. Hence inventory holding cost in OW of these items is given by

$$H\left\{K(W-K)(n+1)/D+(W-K)^2/2D\right\} \quad (7)$$

The fixed order cost per order is A. Hence the total cost of inventory for the system using (2) to (7) is given by

$$C = A+H\left\{\beta W_1Q/R-\beta W_1^2/2R\right\}+F\beta(Q-W_1)^2/2R+FK\beta(Q-W_1)(n+1)/2D$$

$$+nCt+(n+1)HK^2/2D+H\left\{K(W-K)(n+1)/D+(W-K)^2/2D\right\}$$

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Average inventory cost

$$C(Q,K) = C/T$$

Also we have $T = Q/D$, $Z = \beta(Q - W_1)$, $n = Z/K$ and $D = \alpha + \lambda Q^{\delta}$,

$$C(Q,K) = A(\alpha + \lambda Q^{\delta})/Q + \beta FQ/2 + \beta W_1^2(F-H)/2Q - \beta K W_1(F-H)/2Q$$

$$+\beta K(F-H)/2 - \beta W_1(F-H) + \beta C_t(\alpha + \lambda Q^{\delta})(Q - W_1)/QK \tag{8}$$

For optimum value of Q and K, substituting $\delta C(Q,K)/\delta Q = 0$ and $\delta C(Q,K)/\delta K = 0$ $\delta C(Q,K)/\delta Q = O$ implies

$$Q^{\delta+1}\left[\frac{2C_t\lambda\beta}{K}\{\delta - W_1(\delta-1)\}\right] + 2A\lambda(\delta-1)Q^{\delta} + \beta FQ^2$$

$$= 2A\alpha + \beta W_1^2(F-H) - \beta K W_1(F-H) - 2C_t\beta W_1\alpha/K \tag{9}$$

which can not be solved for Q analytically and can be solved with the help of Newton Raphson method.

Also if $\lambda = O$

$$Q^* = \left[2A\alpha/\beta F + W_1^2(1 - H/F) - W_1K(1-H/F) - 2C_tW_1\alpha/KF\right]^{\frac{1}{2}} \tag{10}$$

$\delta C(Q,K)/\delta K = O$ implies

$$K^* = \sqrt{\frac{2C_t(\alpha + \lambda Q^{\delta})}{F-H}}$$

$$\sqrt{\frac{2C_t\alpha}{F-H}} \qquad (\text{If } \lambda = 0) \tag{11}$$

Also from (11) and (10) we have

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$$Q^* = \left[\frac{2A\alpha}{\beta F} + W_1^2(1 - H/F) - \frac{2W_1}{F}\sqrt{2\alpha C_t(F-H)}\right]^{\frac{1}{2}}$$

COST REDUCTION DUE TO *K*-RELEASE RULE

The unit cost of KRR is $C_t' = C_t / K$. Suppose the unit cost of transportation C_t^* without bulk release rule. Bulk transportation is economical only if $C_t^* > C_t'$. Hence without KRR, we have the cost function

$$C(Q) = A(\alpha + \lambda Q^{\delta})/Q + \beta FQ/2 + \beta W_1^2(F-H)/2Q - \beta W_1(F-H)$$

$$+\beta C_t^*(\alpha + \lambda Q^{\delta}) - C_t^*\beta W_1(\alpha + \lambda Q^{\delta})/Q \qquad (13)$$

The optimum value Q' of Q which minimize (13) is obtained by solving

$dC(Q)/dQ = 0$

This implies

$$\beta C_t^*\lambda\delta Q^{\delta+1} + Q^{\delta}\left[A\lambda(\delta-1) - C_t^*\beta W_1\lambda(\delta-1)\right] + \beta FQ^2 = 2A\alpha + \beta W_1^2(F-H) - 2C_t^*\beta W_1\alpha$$

Equation (14) can not be solved for Q analytic and can be solved with the help of Newton Raphson method for obtaining optimal value of Q.

If $\lambda = 0$ then equation (14) reduces to

$$Q_1^1 = \left[2A\alpha/\beta F + W_1^2(F-H)/F - 2C_t^*W_1\alpha/F\right]^{\frac{1}{2}} \qquad (15)$$

KRR is economical if $\left[C(Q) - C(Q,K)\right] > 0$

This implies $\left(C_t^* - C_t'\right) > (F-H)K/2\alpha \qquad (16)$

must be satisfied.

Thus for a given situation inequality (16) is satisfied than KRR will be economical.

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Relation between EOQ with and without KRR.

$$Q^* = \left[(Q')^2 + \frac{2C_t^* \alpha W_1}{F} - \frac{2W_1}{F}\sqrt{2\alpha C_k (F-H)} \right]^{\frac{1}{2}} \tag{17}$$

In general Q^* exceeds Q.

SPECIAL CASES

Case (i) If $\beta \to 1$ i.e. rate of replenishment is infinite.

Then equation (10) for optimum value of Q. becomes

$$Q^* = \left[2A\alpha / F + W^2(1-H/F) - WK(1-H/F) - 2C_t W\alpha / KF \right]^{\frac{1}{2}} \tag{18}$$

Equation (18) is the same as equation (9) as that given by J. Mendus and Antony (8).

Case (ii) If Demand $D = \alpha$ and $\beta \to 1$.

Then equations (10), (11) and (12) of our Model reduces respectively to the equations (6) and (7) of Model (11).

Case (iii) When F is large, Q^* tends to W_1.

Further when $C_t \to 0$ Then $Q^* = \left[2A\alpha / \beta F + W_1^2 (1 - H/F) \right]^{\frac{1}{2}}$

In this case bulk transportation is not necessary.

NUMERICAL EXAMPLE

For a given inventory system, suppose we have $H = 2$, $F = 4$, $A = 100$, $W = 100$, $C_t = 2$, $C_t^* = 0.15$, $\alpha = 2000$, $\lambda = 0$, $R = 6000$, $D = 2000$ Then $\beta = 1 - D/R = 2/3$, $W_1 = W/\beta = 150$.

(a) With KRR, we have $K^* = 63, Q^* = 389$ and $C(Q,K) =$ Rs. 923.17.

(b) Without KRR. $Q = 372$; $C(Q) =$ Rs. 993.31

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Also when $k = 63$, $C_t' = C_t / K = 0.0137$ and $C_t^* = 0.15$ clearly $C_t^* > C_t'$.

Also it is clear $Q^* > Q'$.

CONCLUDING REMARKS

In the present paper, we have formulated and solved a deterministic inventory model for two warehouses considering finite rate of replenishment and stock dependent consumption rate. Profitability of KRR has been desired. Also optimum batch size 'K' is independent of production rate and capacity of own warehouse, but still from the example it is clear that KRR reduces cost considerably. Further it is worth remembering that one should consider the possibility of RW whenever lot size indicated by the classical EOQ model is greater than the capacity of own warehouse. This model further can be extended by considering shortages, as shortages is an unavoidable characteristic of most of the inventory systems implemented in practice.

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OSCILLATION THEOREMS FOR FIRST ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

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INTRODUCTION

In this paper we are concerned with the oscillatory behaviour of first order neutral functional differential equation of the form

$$\frac{d}{dt}\big(x(t)+c(t)x[\tau(t)]\big)+\delta q(t)f\big(x[g(t)]\big)=0, \qquad (1.1;\delta)$$

where $\delta = \pm 1, \tau, c, g, q:[t_0,\infty) \to R = (t_0,\infty)$ and $f: R \to R$ are continuous. The following conditions are always assumed to hold:

(i) $\lim_{t\to\infty} g(t) = \infty$;

(ii) $\tau(t)$ is strictly increasing for $t \geq t_0 \geq 0$ and $\lim_{t\to\infty} \tau(t) = \infty$;

(iii) $xf(x) > 0$ for $x \neq 0$; and

(iv) $q(t) \geq 0$ and not identically zero on any ray of the form $[T,\infty)$ for some $T \geq t_0$.

By a solution of Eq. $(1.1;\delta)$ we mean a function $x:[T_x,\infty)\to R$ such that $x(t)+c(t)x[\tau(t)]$ is continuously differentiable and satisfies Eq. $(1.1;\delta)$ for all large $t \geq T_x$. The solutions which vanish for all large t will be excluded from our consideration.

A solution of Eq. $(1.1;\delta)$ is said to be oscillatory if it has an infinite sequence of zeros tending to infinity, otherwise, a solution is said to be nonoscillatory. Eq.$(1.1;\delta)$ is said to be oscillatory if all its solutions are oscillatory.

The oscillator behavior of first order functional differential equations of neutral type has been intensively studied in the literature in recent years. Most of the literature,

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however, is concerned with linear equations with constant coefficients and constant deviations for which oscillation criteria are given through analysis of the associated characteristic equations, and the references cited therein, and very little is known about the oscillation of Eq. $(1.1;\delta)$ whose coefficients and deviations are variable.

It seems that the known oscillation criteria fail to describe the oscillatory behavior of Eq. $(1.1;\delta)$ when f is not a monotonic function.

In this paper, we deal with the oscillation problem of nonlinear Eq. $(1.1;\delta)$ with variable coefficients and deviations and with f not a monotonic function. We establish some new sufficient conditions for the oscillation of all/and or bounded solutions of Eq. $(1.1;\delta)$ where the function $c(t)$ is either positive or negative for $t \geq t_0$. We also present some new sufficient conditions for the oscillation of the forced neutral equation.

$$\frac{d}{dt}\left(x(t)+c(t)x[\tau(t)]\right)+\delta q(t)f\left(x[g(t)]\right)=e(t), \tag{1.2;δ}$$

where $e:(t_o,\infty)\to R$ is continuous.

Examples are inserted in the text to illustrate the relevance of our theorems.

OSCILLATION OF EQ. (1.1; δ)

The following notations are needed in the sequel:

$$R_{t_o}=(-\infty,-t_o)\cup[t_o,\infty) \quad \text{if} \quad t_o>0$$

$$=(-\infty,0)\cup(0,\infty) \quad \text{if} \quad t_o=0$$

$C(R)=\{f:R\to R: f$ is continuous and $x f(x)>0$ for $x\neq 0\}$.

$C_B(R_{t_o})=\{f\varepsilon C(R): f$ is of bounded variation on any interval $[a,b]\subset R_{t_o}\}$

Suppose $t_o>0$ and $f\varepsilon C(R)$. Then $f\varepsilon C_B(R_{t_o})$ if and only if

$$f(x)=G(x)H(x) \text{ for all } x\in R_{t_o}$$

where

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$G: R_{t_0} \to (0,\infty)$ is nondecreasing on $(-\infty,-t_0)$ and nonicreasing on (t_0,∞) and $H: R_{t_0} \to R$ and nondecreasing on R_{t_0}.

We assume that the functions c and τ are subject to one of the following conditions:

$$0 \le c(t) \le a_1 < 1 \qquad \tau(t) > t; \qquad \text{and} \tag{1.3}$$

$$1 \le b_1 \le c(t) \le b_2 \qquad \tau(t) < t; \tag{1.4}$$

$$0 \le c(t) \le a_1 < 1 \qquad \tau(t) < t; \qquad \text{and} \tag{1.5}$$

$$1 < b_1 \le c(t) \le b_2 \qquad \tau(t) < t. \tag{1.6}$$

We let

$$a = 1 - a_1 \quad \text{and} \qquad b = \frac{b_1 - 1}{b_1 b_2}$$

The main results of this section now follow :

Theorem 1.1 Suppose that $f \varepsilon C(R_{t_0}), t_0 \ge 0$ and let the functions G and H be a pair of continuous components of F, H being the nondecreasing one. Then the following conclusions hold.

(I) Eq. $(1.1;\delta)$ is oscillatory, if condition (1.3) holds and for every positive constant $c<1$, $g(t) < t$ for $t \ge t_0$ and all large t, the first order delay equation

$$y'(t) + aq(t)G(c)H\left(y\left[g(t)\right]\right) = 0 \tag{1.7}$$

is oscillatory.

(II) Eq. $(1.1;1)$ is oscillatory, if condition (1.4) holds and for every positive constant $c<1, \tau^{-1} \circ g(t) < t$ for $t \ge t_0$ (where τ^{-1} denotes the inverse function of $\tau(t)$) and all large t, the first order delay equation

$$v'(t) + bq(t)G(c)H\left(v\left[\tau^{-1} \circ g(t)\right]\right) = 0 \tag{1.8}$$

is oscillatory.

Proof: Let $x(t)$ be a nonoscillatory solution of Eq. $(1.1;1)$. We may assume that $x(t)$

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is eventually positive, since a parallel argument holds if $x(t)$ is eventually negative. Define

$$y(t) = x(t) + c(t)x[\tau(t)], \tag{1.9}$$

then $y(t) > 0$ for $t \geq t_0$ and Eq. $(1.1;\delta)$ implies that $y(t)$ is a decreasing function for all large t. Using this fact, we see from (1.9) that if condition (1.3) holds, then

$$\begin{aligned} x(t) &= y(t) - c(t)x[\tau(t)] \\ &= y(t) - c(t)\big[y[\tau(t)] - c[\tau(t)]x[\tau \circ \tau(t)]\big] \\ &\geq (1 - c(t))y(t) \\ &\geq (1 - a_1)y(t), \end{aligned}$$

or

$$x(t) \geq ay(t), \qquad \text{for all large } t, \tag{1.10}$$

and if condition (1.4) holds, then

$$\begin{aligned} x(t) &= \frac{1}{c[\tau^{-1}(t)]}\big(y[\tau^{-1}(t)] - x[\tau^{-1}(t)]\big) \\ &= \frac{y[\tau^{-1}(t)]}{c[\tau^{-1}(t)]} - \frac{1}{c[\tau^{-1}(t)]}\left(\frac{y[\tau^{-1} \circ \tau^{-1}(t)]}{c[\tau^{-1} \circ \tau^{-1}(t)]} - x\frac{[\tau^{-1} \circ \tau^{-1}(t)]}{c[\tau^{-1} \circ \tau^{-1}(t)]}\right) \\ &\geq \frac{c[\tau^{-1} \circ \tau^{-1}(t)] - 1}{c[\tau^{-1}(t)]c[\tau^{-1} \circ \tau^{-1}(t)]}y[\tau^{-1}(t)] \\ &\geq \left(\frac{b_1 - 1}{b_1 b_2}\right)y[\tau^{-1}(t)], \end{aligned}$$

or

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$$x(t) \geq b \qquad y\left[\tau^{-1}(t)\right], \quad \text{for all large } t. \tag{1.11}$$

(A) Suppose that condition (1.3) holds. Then from (1.10) and the fact that $y'(t) \leq 0$ for all large t, it follows that there exist a $T \geq t_0$ and a positive constant $c^* < 1$ such that for all $t \geq T$

$$x[g(t)] \leq y[g(t)] \leq c^* \tag{1.12}$$

and

$$x[g(t)] \geq ay[g(t)]. \tag{1.13}$$

Using (1.12) and (1.13) in Eq. (1.1:δ), we have

$$\begin{aligned} 0 &= y'(t) + q(t) f\left(x[g(t)]\right) \\ &= y'(t) + q(t) G\left(x[g(t)]\right) H\left(x[g(t)]\right) \\ &\geq y'(t) + G\left(c^*\right) q(t) H\left(ay[g(t)]\right), \end{aligned}$$

or

$$w'(t) + aG\left(c^*\right) q(t) H\left(w[g(t)]\right) \leq 0 \text{ for } t \geq T, \tag{1.14}$$

where $w(t) = ay(t)$. It is easy to check that for all $t \geq T$

$$w(t) \geq aG(c) \int_t^{\infty} q(s) H\left(w[g(s)]\right) ds.$$

The function $w(t)$ is positive and strictly decreasing for $t \geq T$.

(B) Suppose that condition (1.4) holds. Then from (1.11) and the fact that $y'(t) \leq 0$ for all large t, there exist a positive constant $c^* < 1$ and $aT^* \geq t_0$ such that (1.12) holds and

$$x[g(t)] \geq b \quad y\left[\tau^{-1} \circ g(t)\right] \qquad \text{for} \quad t \geq T^*. \tag{1.15}$$

Now, using (1.12) and (1.15) in Eq. (1.1;δ), we have,

$$y'(t) + G\left(c^*\right) q(t) H\left(by\left[\tau^{-1} \circ g(t)\right]\right) \leq 0 \text{ for } t \geq T^*,$$

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or

$$w'(t) + bG(c^*)q(t)H\left(w\left[\tau^{-1} \circ g(t)\right]\right) \le 0 \text{ for } t \ge T^*. \tag{1.16}$$

Where $w(t) = by(t)$.
Proceeding as in the above case, we obtain the desired contradiction.
This completes the proof.

For any continuous function, $\sigma:[t_0,\infty) \to R$ with $\lim_{t\to\infty} \sigma(t) = \infty$,

we define

$$R(\sigma) = \{t \in [t_0,\infty]: t_0 \le \sigma(t) < 1\}.$$

Also, we assume that

$$H(x)sgn \ge |x|^{\alpha} \text{ for } x > 0 \text{ and } \alpha > 0. \tag{1.17}$$

As an application of Theorem 1.1, we get the following oscillation criterion for Eq. $(1.1;\delta)$.

Corollary 1.1 Suppose that $f \in C(R_{t_0}), t_0 \ge 0$ and let the functions G and H form a pair of continuous components of F,H being the nondecreasing one. Then,

(I) Eq. $(1.1;\delta)$ is oscillatory if conditions (1.3) and (1.17) hold and one of the following conditions (A_1) or (B_1) holds:

(A1) $\alpha = 1$, and

$g(t) < t$ and $g(t)$ is nondecreasing for $t \ge t_0$; (1.18)

and for every positive constant $c < 1$,

$$\lim_{t\to\infty} inf \int_{g(t)}^{t} q(s)ds > \frac{1}{aeG(c)}. \tag{1.19}$$

(B_1) $a < 1$, and

$$\int_{R(g)} q(s)ds = \infty. \tag{1.20}$$

(II) Eq. $(1.1;\delta)$ is oscillatory if conditions (1.4) and (1.17) hold and one of the following conditions (C_1) or (D_1) holds :

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(C_1) $\alpha = 1$, and

$\sigma(t) = \tau^{-1} \circ g(t) \le t$ and $\sigma(t)$ is nondecreasing for $t \ge t_0$ and (1.21)

for every positive constant $c<1$,

$$\lim_{t\to\infty} \inf \int_{\sigma(t)}^{t} q(s)ds > \frac{1}{beG(c).} \tag{1.22}$$

(D_1) $\alpha < 1$ and

$$\int_{R(\tau^{-1}\circ g)} q(s)ds = \infty. \tag{1.23}$$

Proof: Let $x(t)$ be an eventually positive solution of Eq. (1.1;δ). As in the proof of Theorem 1.1, we obtain the inequalities (1.14) and (1.17). Now, we consider the inequality (1.14).

(A_1) From (1.17), inequality (1.14) takes the form

$$w'(t) + aG(c^*)q(t)w[g(t)] \le 0 \text{ for } t \ge T_1 \ge T. \tag{1.24}$$

In view of condition (1.19), it follows that inequality (1.24) cannot have an eventually positive solution. This contradicts the fact that $w(t) = a\,y(t) > 0$ eventually.

(B_1) Since $w[g(t)] \ge w(t)$ for all $t \in R(g) \cap [T, \infty)$, we have

$$\int_{R(g)\cap[T,\infty]} q(s)ds \le \frac{1}{aG(c^*)} \int_0^{w(T)} v^{-\alpha} dv$$

$$= \frac{1}{aG(c^*)} \left[\frac{w^{1-\alpha}(T)}{1-\alpha} \right] < \infty,$$

which contradicts (1.20)

Next, we consider the inequality (1.16). By a similar argument, as above, we obtain the desired conclusion.

The following example is illustrative.

Example 1.1 Each of the following neutral equations

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$$\frac{d}{dt}\left(x(t)+\frac{1}{2}x[t+2\pi]\right)+\frac{3}{2}e^{\sin^2 t}\,e^{-x^{2[t-3\pi]}}x[t-3\pi]=0, t\geq 0, \tag{1.25}$$

and

$$\frac{d}{dt}\left(x(t)+2x[t-2\pi]\right)+3e^{|\sin t|}e^{-|x[t-5\pi]|}x[t-5\pi]=0, t\geq 0, \tag{1.26}$$

has an oscillatory solution $x(t)=sint.$

All conditions of corollary 1.1 (I-A_1) and (II-C_1) are satisfied for Eq. (1.25) and Eq. (1.26) respectively.

BOUNDED OSCILLATION OF EQ. (1.1; δ)

In this section we are concerned with the oscillatory behavior of all bounded solutions of Eq. (1.1; δ) when condition (1.5) or (1.6) holds. Also we study the oscillatory behavior of Eq. (1.1) where $c(t)$ is negative.

Theorem 1.2 Suppose that $f \in C\left(R_{t_0}\right), t_0 \geq 0$ and let the functions G and H form a pair of continuous components of F,H being the nondecreasing one. Then,

(I) All bounded solutions of Eq. (1.1;-1) are oscillatory if condition (1.5) holds and

one of the following conditions (A) or (B) is satisfied:

(A)

$$\int_0^\infty q(s)ds=\infty \tag{1.27}$$

(B) Condition (1.16) holds with $\alpha=1$,

$$g(t)\geq t \text{ and } g(t) \text{ is nondecreasing for } t\geq t_0, \text{ and for every positive constant } c\geq 1, \tag{1.28}$$

$$\liminf_{t\to\infty}\int_t^{g(t)} q(s)ds > \frac{1}{aeG(c)}. \tag{1.29}$$

(II) All bounded solutions of Eq. (1.1; δ) are oscillatory if condition (1.6) holds and one of the following conditions (C) or (D) is satisfied

(C) Condition (1.27) holds;

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(D) Condition (1.17) holds with $\alpha = 1$,

$\sigma(t) = \tau^{-1} \circ g(t) \geq t$ and $\sigma(t)$ is nondecreasing for $t \geq t_0$, and for every positive constant $c \geq 1$, (1.30)

$$\liminf_{t\to\infty} \int_{t}^{\sigma(t)} q(s)ds > \frac{1}{beG(c)}. \tag{1.31}$$

Proof: Let $x(t)$ be a bounded and eventually positive solution of Eq. $(1.1;\delta)$. Define $y(t)$ by (1.9). Then $y(t)$ is eventually positive. From Eq. $(1,1;\delta)$ it follows that y (t) is nondecreasing for all large t, and it follows that (1.10) or (1.11) is satisfied according as condition (1.5) or (1.6) holds. Since $x(t)$ is bounded and $y(t)$ is nondecrasing for $t \geq t_0$, there exist positive constants $\beta_1 \geq 1$ and $\beta_2 > 0$ and a $T \geq t_0$ such that for $t \geq T$

$$x[g(t)] \leq \beta_1 \quad \text{and} \quad |y[g(t)]| > \beta_2 \tag{1.32}$$

and both (1.13) and (1.15) hold.

(I) Assume that condition (1.5) holds. Inview of (1.13) and (1.32), it follows from Eq. (1.1-1)

$$y'(t) \geq qG(\beta_1)H(a\beta_2) \quad \text{for} \quad t \geq T. \tag{1.33}$$

Integrating (1.33) from T to t, we obtain

$$\infty > y(t) \geq y(T) + G(\beta_1)H(a\beta_2)\int_{T}^{t} q(s)ds$$

$$\to \infty \text{ as } t \to \infty,$$

a contradiction.

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Next, using (1.17) with $\alpha = 1$ and (1.32) in Eq. $(1.1;\delta)$ we have

$$y'(t) - aG(c_1)q(t)y[g(t)] \geq 0 \quad \text{for} \quad t \geq T. \tag{1.34}$$

But, inview of condition (1.29), it follows that inequality (1.31) cannot have an eventually positive solution. This contradicts the fact that $y(t)$ is eventually positive.

(II) Assume that condition (1.5) holds. Inview of (1.15) and (1.32), it follows from Eq. $(1.1;\delta)$ that

$$y'(t) \geq G(\beta_1)q(t)H(by[\sigma(t)]) \quad \text{for} \quad t \geq T.$$

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Proceeding as in the proof of the above case, we obtain the desired conclusion. This completes the proof.

For illustration we consider the following example:

Example:1.2 Consider the neutral equations

$$\frac{d}{dt}\left(x(t)+\frac{1}{2}x[t-\sigma]\right)=\left(2+e^{-\sigma}\right)e^{t}\cosh t^{2}\left(\frac{x\left[t^{2}\right]}{1+x^{2}\left[t^{2}\right]}\right)=0 \qquad t\geq 0, \tag{1.35}$$

and

$$\frac{d}{dt}\left(x(t)+2x[t+\sigma]\right)=\left(2e^{-\sigma}+e^{-2\sigma}\right)\cosh\left(e^{t+2\sigma}\right) \tag{1.36}$$

$$\operatorname{sech}\left(x[t+2\sigma]\right)x[t+2\sigma]=0, \qquad t\geq 0,$$

where σ is a positive constant, $\sigma \geq 2$.

All conditions of Theorem 1.2 (1-B) and(II-D) are satisfied for Eq. (1.35) and (1.36) respectively, and hence all bounded solutions of Eq. (1.35) and Eq. (1.36) are oscillatory.

We note that each of the equation (1.35) and (1.36) has an unbounded nonoscillatory solution $x(t)=e^{t}$.

Next, we present two criteria for the oscillation of all bounded solutions of Eq.$(1.1;\delta),\delta=\pm 1$, when the function c (t) is assumed to be negative.

We assume that

$$0<-c(t)=\beta(t)<\beta_{0} \text{ for some positive constant } \beta_{0}. \tag{1.37}$$

Theorems. Let $f\varepsilon\, C\left(R_{t_0}\right)$, $t_0\geq 0$ and conditions (1.18) and (1.37) hold and suppose that the functions G and H form a pair of continuous components of F,H being the nondecreasing one. Suppose that for every positive constant c and all large t the delay first orders equation

$$z'(t)+q(t)G(c)H\left(z[g(t)]\right)=0 \tag{1.38}$$

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is oscillatory, Then every bounded solution of Eq. (1:1;1) is oscillatory provided that either condition (1.27) is satisfied or in addition to conditions (1.17) with $\alpha = 1$ and (1.30) the condition

$$\liminf_{t\to\infty} \int_{t}^{\sigma(t)} q(s)ds > \frac{1}{e\beta_0 G(C)} \tag{1.39}$$

is satisfied.

Proof: Let $x(t)$ be a bounded and eventually positive solution of Eq. (1.1;1).

Define

$$y(t) = x(t) - \beta(t)x[\tau(t)]. \tag{1.40}$$

Then, inview of Eq. (1.1:δ).

$y'(t) = -q(t)f\left(x[g(t)]\right) \le 0$ eventually and hence $y(t)$ is nonicreasing for all large. t Thus, $y(t)$ is of one sign for all large t, say $t \ge t_1 \ge t_0$.

We consider the following two cases:

Case 1: Let $y(t) > 0$ for $t \ge t_1$. There exist a positive constant γ and a $t_2 \ge t_1$ such that for all $t \ge t_2$.

$$x[g(t)] \le \gamma \tag{1.41}$$

and

$$x[g(t)] \ge y[g(t)]. \tag{1.42}$$

Consequently Eq. (1.1;1) reduces to

$$y'(t) + G(\gamma)q(t)H\left(y[g(t)]\right) \le 0 \text{ for } t \ge t_2.$$

Proceeding as in the proof of Theorem 1.1, we obtain the desired contradiction.

Case 2: Let $w(t) < 0$ for $t \ge t_1$. Using (1.41), we get

$$0 < -y(t) = u(t) = \beta(t)x[\tau(t)] - x(t)$$

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$$\leq \beta(t)x[\tau(t)]$$

$$\leq \beta_0(t)x[\tau(t)] \qquad \text{for} \quad t \geq t_2$$

and here,

$$x(t) \geq \frac{1}{\beta_0} u[\tau^{-1}(t)] \text{ for } t \geq t_3 \geq t_2$$

From Eq. (1.1;δ) it follows that

$$w'(t) \geq \frac{1}{\beta_0} G(\gamma) q(t) H\left(w[\sigma^{-1} \circ g(t)]\right) \text{ for } t \geq t_3,$$

where $u(t)=\beta_0 w(t)$.

The rest of the proof is similar to that of Theorem 1.2 and hence is omitted. This completes the proof.

Theorem 1.4 Let $f \varepsilon C(R_{t_0})$, $t_0 \geq 0$, conditions (1.21) and (1.37) hold, and let the functions G and H form a pair of continuous components of F, H being the nondecreasing one. If for every positive constant c and all large t the delay first order equation.

$$z'(t) + \frac{1}{\beta_0} G(c) q(t) H\left(z[\tau^{-1} \circ g(t)]\right) = 0 \tag{1.43}$$

is oscillatory, then every bounded solution of Eq. (1.1;-1) is oscillatory provided that either condition (1.27) holds, or in addition to condition (1.17) with $\alpha = 1$ and (1.28) the condition.

$$\liminf_{t \to \infty} \int_t^{g(t)} q(s)ds > \frac{1}{eG(c)} \text{ holds.} \tag{1.44}$$

Proof : Let $x(t)$ be a bounded an eventually positive solution of Eq. (1.1;1). Define $y(t)$ by (1.40). Then, it follows that $y(t)$ is nondecreasing and is of one sign for all large t. Now, we consider the following two cases:

Case 1: Suppose that $y(t) > 0$ for $t \geq t_1 \geq t_0$. Then there exist a $t_2 \geq t_1$ and a positive constant γ such that

$x[g(t)] \leq \gamma$ and $x[g(t)] \geq w[g(t)]$, for $t \geq t_2$,

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and hence

$$y'(t) \geq G(\gamma)q(t)H\big(y[g(t)]\big) \quad \text{for } t \geq t_2.$$

The rest of the proof is similar to that of Theorems 1.2 (I) and hence is omitted.

Case 2. Suppose that $y(t) < 0$ for $t \geq t_1 \geq t_0$. Set

$$0 > -y(t) = u(t)$$

and proceeds as in the proof of Theorem 1.3-case 2 to obtain

$$v'(t) + \frac{1}{\beta_0} G(\gamma)q(t)H\big(v[\tau^{-1} \circ g(t)]\big) \geq 0 \quad \text{for } t \geq t_2,$$

where $u(t) = \beta_0 v(t)$. We use an argument similar to the one used in the proof of Theorem 1.1 (II) to complete the proof.

The following examples are illustrative:

Example 1.3 The first order neutral equation

$$\frac{d}{dt}\big(x(t) - \gamma x[t+2\pi]\big) + \left(\frac{1}{t^2} - \frac{\gamma}{(t+2\pi)^2}\right)\left(\frac{1+(t-\pi)^2}{t-\pi}\right) \tag{1.45}$$

$$\bullet \frac{x[t-x]}{1+x^2[t-\pi]} = 0, \; t > \pi + y,$$

has a bounded nonoscillatory solution $x(t) = t$. Here γ is any positive constant. All conditions of Theorem 1.3 are satisfied the equation

$$\frac{d}{dt}y(t) + \left(\frac{1}{1+c^2}\right)\left(\frac{1}{t^2} - \frac{\gamma}{(t+2\pi)^2}\right)\left(\frac{1+(t-\pi)^2}{t-\pi}\right)y(t-\pi) = 0,$$

for $\gamma > 0$ and $t > \pi$,

has a positive solution $y(t)$ with $\lim_{t\to\infty} y(t) = 0$ for every positive constant c.

Example 1.4 Consider the neutral equation

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$$\frac{d}{dt}\left(x(t)-\gamma x\left[t^2\right]\right)+(2\gamma t-1)\left(t^{-\frac{1}{6}}+t^{\frac{5}{6}}\right)\left(\frac{x^{\frac{1}{3}}\left[\sqrt{t}\right]}{1+x^2\left[\sqrt{t}\right]}\right)=0, t>\frac{1}{2\gamma}, \tag{1.46}$$

where γ is any positive constant. It is easy to check that for any positive constant c, the equation

$$y'(t)+\frac{1}{1+c^2}(2\gamma t-1)\left(t^{-\frac{1}{6}}+t^{\frac{5}{6}}\right)y^{\frac{1}{3}}\left[\sqrt{t}\right]=0$$

is oscillatory (see Corollary 1.1). Therefore, all conditions of Theorem 1.3 are satisfied and hence all bounded solutions of Eq. (1.46) are oscillatory. Note that Eq. (1.46) has an unbounded nonoscillatory solution $x(t)=t$.

Example 1.5 The first order neutral equation

$$\frac{d}{dt}(x(t)-2x[t-1])=\left(\frac{2e-1}{e^2}\right)\left(1+e^4e^{-2t}\right)\left(\frac{x[t-2]}{1+x^2[t-2]}\right)=0, t>0 \tag{1.47}$$

has a bounded nonoscillatory solution $x(t)=e^{-t}$. Here $\tau(t)=t-1, g(t)=t-2$ and hence $\tau^{-1}\circ g(t)=t-1>t$.

All conditions of Theorem 1.4 are satisfied, the delay equation,

$$y'(t)+\frac{1}{2\left(1+c^2\right)}\left(\frac{2e-1}{e^2}\right)\left(1+e^4e^{-2t}\right)y[t-1]=0, t>0$$

has a positive solution $y(t)$ with $\lim_{t\to\infty} y(t)=0$.

1.4. Oscillation of Eq. (1.2; δ)

In this section we establish some sufficient conditions under which Eq. (1.2;δ), $\delta=\pm 1$ is oscillatory.

We list the assumptions which are needed below :

$$0>c(t)=a_1>1, \tau(t)=t-\sigma, \quad \sigma \text{ is a positive constant;} \tag{1.48}$$

$$c(t)=b_1=b_2>1, \tau(t)=t+\sigma, \quad \sigma \text{ is a positive constant.} \tag{1.49}$$

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Also, we assume the following conditions on the forcing term:

There exists a differentiable and oscillatory function

$$\eta:[t_0,\infty] \to R \text{ Such that } \eta'(t) = e(t), \tag{1.50}$$

and

$\eta(t)$ is periodic of period σ i.e. (1.51)

$\eta(t \pm \sigma) = \eta(t)$, where σ is defined as in (1.49) and (1.50).

Theorem 1.5. Let condition (1.50) hold, $c(t) = a$, a is a nonnegative constant and $\tau(t) = t + \delta\sigma$, $\delta = \pm 1$, and σ is a positive constant. If

$$\text{(1.52)} \qquad \limsup_{t\to\infty} \eta(t) = +\infty \quad \text{and} \quad \liminf_{t\to\infty} \eta(t) = -\infty,$$

then,

(i) All bounded solutions of Eq. $(1.2;\delta)$ are oscillatory,

(ii) Eq. $(1.2;\delta)$ is oscillatory.

Proof: Let $x(t)$ be a bounded an eventually positive of Eq.$(1.2;\delta)$ and $x(t)$ be an eventually positive solution of Eq. $(1.2;\delta)$ Define

$$y(t) = x(t) + ax[t + \delta_1\sigma] \text{ and } \quad \tau(t) = y(t) - \eta(t), \tag{1.53}$$

where $\delta_1 = \pm 1$.

Then Eq. (1.3;) take the form

$$z'(t) = -\delta q(t) f(x[g(t)]). \tag{1.54}$$

Next, we consider the following two cases:

Case 1. $\delta_1 = -1$. It follows that $z(t)$ is an increasing function, say on $[t_1\infty)$ for some $t_1 \geq t_0 \geq 0$. We show that $z(t) > 0$ for $t \geq T \geq t_1$.

If not, then $z(t) < 0$ for $t \geq T \geq t_1$. Hence

$$y(t) - \eta(t) > 0 \text{ i.e } 0 > y(t) < \eta(t) \text{ for } t \geq T,$$

which is a contradiction, since $\eta(t)$ is oscillatory. Thus, we have

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$$z(t) > 0 \text{ and } z'(0) > 0 \quad \text{for} \quad t \geq T. \tag{1.55}$$

Taking *lim* superior in $y(t) > \eta(t)$ as $t \to \infty$, we obtain

$$\limsup_{t\to\infty} y(t) > \limsup_{t\to\infty} \eta(t) = \infty$$

which contradicts the fact that $y(t)$ is bounded.

Case 2. $\delta_1 = 1$. From (1.54), it follows that $z(t)$ is an eventually of one sign. As in the proof of case 1, above, we see that

$$z(t) > 0 \text{ and } z'(t) < 0 \quad \text{for} \quad t \geq T. \tag{1.56}$$

Since $z(t) + \eta(t) = y(t) > 0$, we have

$z(t) > -\eta(t)$, from which it follows that

$$\limsup_{t\to\infty} z(t) \geq \limsup_{t\to\infty} (-\eta(t)) = -\liminf_{t\to\infty} \eta(t) = \infty$$

which contradicts the fact that $z(t)$ is bounded from above. This completes the proof.

For illustration we provide the following two examples:

Example 1.6 Consider the forced neutral equation

$$\frac{d}{dt}\left(x(t) + ax[t+\delta\sigma]\right) - \frac{\left(1+ae^{-\delta\sigma}\right)e-t}{\left(1-e^{-g(t)}\right)^{\gamma}}\left(\left(1+\left(1-e^{-g(t)}\right)^2\right)\right) \tag{1.57}$$

$$\cdot\frac{\left|x[g(t)]\right|^{\gamma}}{1+x^2[g(t)]} sgn\, x[g(t)] = t.cost + sin\ t,\ t>0\ ,$$

where $\delta = \pm 1$, a and σ are nonnegative real numbers, α is a positive constant, g: $[t_0, \infty] \to R$ is continuous and $\lim_{t\to\infty} g(t) = \infty$. If we choose $\eta(t) = t.sint$, then all the hypotheses of Theorem 1.5 (i) are satisfied and hence every bounded solution of Eq. (1.57) is oscillatory, It is easy to verify that the corresponding unforced neutral equation, namely

$$\frac{d}{dt}\left(x(t) + ax[t+\delta\sigma]\right) - \frac{\left(1+ae^{-\delta\sigma}\right)}{\left(1-e^{-g(t)}\right)^{\alpha}}\left[1+\left(1-e^{-g(t)}\right)^2\right] \tag{1.58}$$

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$$\bullet\frac{|x[g(t)]|^{\alpha}}{1+x^{2}[g(t)]}\,sgn\,x[g(t)]=0, t>0,$$

has a bounded nonoscillatory solution $x(t)=1-e^{-t}$.

Example 1.7 Consider the forced neutral equation

$$\frac{d}{dt}\left(x(t)+ax[t+\delta\sigma]\right)+\left(1+ae^{-\delta\sigma}\right)\left(e^{\alpha g(t)-t}\right) \tag{1.59}$$

$$\bullet\left(1+e^{-g(t)}\right)\left(\frac{|x[g(t)]|^{\alpha}}{1+|x[g(t)]|}\right)sgn\,x[g(t)]=e^{t}(sin\,t+cost),$$

where $\delta=\pm 1$, a and σ are nonnegative constants, $\alpha>0$, and $g:(t_0,\infty)\to R$ is continuous and $\lim_{t\to\infty}g(t)=\infty$. Here, we choose $\eta(t)=e^{t}\,sin\,t$ and find that all the conditions of Theorem 1.5 (ii) are fulfiled. Thus, Eq. (1.59) is oscillatory we also note that the corresponding unforced neutral equation

$$\frac{d}{dt}\left(x+ax[t+\delta\sigma]\right)+\left(1+ae^{-\delta\sigma}\right)\left(1+e^{-g(t)}\right) \tag{1.60}$$

$$\left(\frac{|x[g(t)]|^{\alpha}}{1+|x[g(t)]|}\right)sgn\;x[g(t)]=0, \qquad t>0,$$

has a nonoscillatory solution $x(t)=e^{-t}$.

Theorem 1.6 Let $f\;\varepsilon\;C\left(R_{t_0}\right), t_0\geq 0$ and conditions (1.50) and (1.51) hold.

Suppose that G and H form a pair of continuous components of F, H being the nondecreasing one. Then the following conclusions hold:

(I) All bounded solutions of Eq. $(1.2;\delta)$ are oscillatory if condition (1.48) holds and either

(A) Condition (1.27) holds; or

(B) In addition to condition (1.7) with $\alpha=1$ and (1.28) the condition (1.29) holds for every constant $c\geq 1$.

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(II) All bounded solutions of Eq. $(1.2;\delta)$ are oscillatory if condition (1.49) holds and either:

(C) Condition (1.27) holds: or

(D) In addition to conditions (1.17) with $\alpha = 1$ and (1.30) the condition (1.31) holds for every constant $c \geq 1$.

Proof: Let $x(t)$ be a bounded and eventually positive solution of Eq. $(1.2;\delta)$. As in the proof of Theorem 1.5, we define the functions *y (t) and z* (t) as in (1.53) and obtain

$$z(t) > 0 \quad \text{and} \quad z'(t) > 0 \text{ eventually.} \tag{1.61}$$

Next, we consider:

(I) Let condition (1.48) hold. From (1.50), (1.51), (1.53) with $\delta_1 = -1$ and (1.61), we see that

$$\begin{aligned} x(t) &= z(t) + \eta(t) - a_1 x[t-\sigma] \\ &= z(t) + \eta(t) - a_1\big[z(t-\sigma) + \eta[t-\sigma] - a_1 x[t-2\sigma]\big] \\ &\geq a\big(z(t) + \eta(t)\big) \text{ for all large } t, \end{aligned}$$

where $a = 1 - a_1$. Next by (1.51), there exists a $T \geq t_0$ *(say)* such that for all $t \geq T$

$$x(t) \geq a\big(z(t) + \eta(T)\big) = w(t).$$

Clearly $az'(t) = w'(t)$ and $w(t) > 0$ for all $t \geq T$. Since $x(t)$ is bounded, there exist a $T_1 \geq T$ and a positive constant γ such that for $t \geq T_1$

$$x[g[t]] \leq \gamma \quad \text{and} \quad x[g(t)] \geq w[g(t)]. \tag{1.62}$$

Thus, $$w'(t) \geq aq(t)G(\gamma)H\big(w[g(t)]\big) \text{ for all } t \geq T_1.$$

The rest of the proof is similar to the proof of Theorem 1.2 (I) and hence is omitted.

(II) Let condition (1.49) hold. From (1.50), (1.53) with $\delta_1 = 1$ and (1.61), we obtain

$$x(t) = \frac{1}{b_1}\big(z[t-\sigma] + \eta[t-\sigma] - x[t-\sigma]\big)$$

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$$= \frac{1}{b_1}\left(z[t-\sigma]+\eta[t-\sigma]\right)-\frac{1}{b_1^2}\left(z[t-2\sigma]+\eta[t-2\sigma]-x[t-2\sigma]\right)$$

$$\geq \left(\frac{b_1-1}{b_1^2}\right)\left(z[t-\sigma]+\eta[t-\sigma]\right)$$

$$= b\left(z[t-\sigma]+\eta[t-\sigma]\right) \text{ for all large } t,$$

where $b=\dfrac{b_1-1}{b_1^2}$.

Once again by (1.51), there exists a $T^* \geq t_1+\sigma$ for some $t_1 \geq t_0$ Such that for $t \geq T^*$ we have

$$x(t) \geq b\left(z[t-\sigma]+\eta\left[T^*-\sigma\right]\right) = v[t-\sigma],$$

and $bz'(t) = v'(t)$ and $v(t) > 0$ for $t \geq T^*$. Since $x(t)$ is bounded there exist a constant $\gamma_1 \geq 1$ and a $T_1^* \geq T^*$ such that

$$x[g(t)] \leq \gamma_1 \text{ and } x[g(t)] \geq v[g(t)-\sigma] \text{ for all } t \geq T_1^* \tag{1.63}$$

Thus $$v'(t) \geq bG(\gamma_1)q(t)H\left(v[g(t)-\sigma]\right) \text{ for } t \geq T_1^*.$$

Proceeding as in the proof of Theorem 1.2 (II), we obtain the desired conclusion and the proof is now complete.

For illustration we consider the following two examples:

Example 1.8 Consider the forced neutral equation

$$\frac{d}{dt}\left(x(t)+cx[t-2\pi\delta_1]\right) = ce^{|\cos t|}\left(e^{-|x[g(t)]|}\right) \cdot \tag{1.64;δ_1}$$

$$\left(\left|x[q(t)]\right|^{\alpha}\right) sgn\ x[g(t)] + cost,\ t>0$$

where $\alpha > 0, \delta_1 = \pm 1$ and c is a positive constant such that

(i) $\delta_1 = 1$ if $0<c<1$; *and*

(ii) $\delta_1 = -1$ if $c>1$,

$g:[t_0,\infty) \to R$ is continuous and $\lim_{t\to\infty} g(t) = \infty$.

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Here, we choose the functions, g, H and η as follows

$$G(x) = e^{-|x|}, H(x) = |x|^{\alpha} \; sgn \; x \text{ and } \eta(t) = Sin\, t.$$

Then $\eta(t)$ is oscillaory and periodic of period 2π. Also,

$$\eta'(t) = cost = e(t).$$

All conditions of Theorem 1.6 (I-A) and (II-C) are satisfied for (i) and (ii) respectively and hence all bounded solutions of Eq. $(1.64;\delta_1)$ are oscillatory.

We observe that if $\alpha = 1$ and $g(t) = t - \dfrac{3\pi}{2}$, Eq. $(1.64;\delta_1)$ has a bounded and oscillatory solution $x(t) = sint$.

Example. 1.9 Consider the forced neutral equation

$$\frac{d}{dt}\left(x(t) + cx[t - 2\pi\delta_1]\right) = \left(\frac{1 + c + c\cos t}{\tanh(t + \sin t)}\right) \qquad (1.65;\delta)$$

$$sech\left(x[g(t)]\right)\sinh\left(x[g(t)]\right) + \cos t, \; t > 0$$

where $\delta_1 = \pm 1$, c and g are defined as in Eq. $(1.164;\delta_1)$ and (i) and (ii) in Example 1.8 hold. Here, we let

$G(x) = sech\, x$, $H(x) = sinh\, x$ and $\eta(t) = sint$.

All conditions of Theorem 1.6 (I.A.) and (II.C) are satisfied for (i) and (ii) hold respectively, and hence every bounded solution of Eq. $(1.65;\delta)$ is oscillatory, It is easy to check that if $g(t) = t$, Eq.$(1.6;\delta_1)$ has an unbounded nonoscillatory solution x $x(t) = t + sint$, $t > 0$.

SOME GENERAL REMARKS

1. We note that if $f(x)$ in Eq. $(1.;\delta_1)$ and $(1.2;\delta_1)$, $\delta_1 = \pm 1$, is a nondecreasing function for $x \neq 0$ (i.e.,the components $G(x)$ of $F(x)$ is identically 1), then the boundedness requirement on the solution of Eq. $(1.1.\delta)$ and Eq. $(1.2;\delta)$ in Theorems 1.2, 1.3, 1.4 and 1.6 can be removed.

2. The function $f(x)$ in Eq. $(1.2;\delta)$, $\delta_1 = \pm 1$ may not satisfy the linearization conditions. For example, we see that if

$$\frac{f(x)}{x} = \frac{|x|^{\alpha-1}}{1 + x^2} \to 1 \qquad \text{as} \qquad x \to 0, \qquad \alpha = 1$$

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$$\to 1 \quad \text{as} \quad x \to \pm\infty, \quad \alpha = 3$$

$$\to \infty \quad \text{as} \quad x \to \pm\infty, \quad \alpha = 3.$$

The lineariztion conditions are satisfied while for the case when

$$\frac{f(x)}{x} = e^{-|x|}|x|^{\alpha-1} \to 0 \text{ as } x \to 0 \text{ and } x \to \pm\infty, \quad \alpha > 1.$$

The linearization conditions fail. Thus, the results of this paper are applicable to a larger class of equations of type $(1.1;\delta_1)$ and $(1.2;\delta_1)$, $\delta_1 = \pm 1$.

3. The forcing term $e(t)$ in Eq. $(1.1;\delta_1)$ and the function η defined by (1.50) and (1.51) are assumed to be oscillatory and periodic and hence bounded for $t \geq t_0$. We note that:
   (i) The forcing term e considered in Eq. $(1.2;\delta_1)$ is not necessarily small.
   (ii) From Examples 1.6 and 1.7 it is evident that the presence of a forcing term can generate oscillations in an otherwise nonoscillatory equation.

Finally, it should be interesting to obtain results similar to Theorems 1.2-1.6 for Eq.(1.1;-1) and Eq. (1.2;-1) without the boundedness restriction on the solutions under consideration.

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PROPAGATION OF STRONG SPHERICAL DIVERGING SHOCK WAVE AND CHANGE IN ENTROPY OF NON-UNIFORM MEDIUM

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ABSTRACT

Chester-Chisnell-Whitham method has been used to study the propagation of shperical diverging strong shock wave in non-uniform medium. Assuming an initial density distribution law as $\rho_0 = \rho' r^{-\omega}$ where ρ' is the density at the centre and ω is a constant, analytical relation for shock velocity has been derived. The expressions for pressure, particle velocity, sound velocity, temperature and change in entropy of the medium immediately behind the shock have been obtained. Finally, the results accomplished here have also been compared with earlier results as well as with those for converging shock. It is found that change in entropy increases as shock propagates in the medium. This agrees with the universal law of increase of entropy.

Key words : Diverging shock wave (Non-uniform medium), Change in entropy

INTRODUCTION

On account of its immense importance in Astrophysics and Thermodynamics, propagation of shock wave (plane, cylindrical, and spherical) has received considerable attention of many workers [1-10]. Considering the effect of overtaking disturbances, the propagation of strong diverging shock in non-uniform medium, the temperature variation has been computed by Yadav and Rana [11].

Yadav *et al* [12-13] have studied the propagation of spherical converging strong shock in uniform medium and obtained the change in entropy and temperature of non-uniform medium on propagation of spherical converging strong shock wave.

Recently, Yadav and Gangwar [14] have studied the propagation of spherical converging strong shock in non-uniform medium. In the present paper CCW method has been used to study the propagation of spherical diverging strong

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shock in non-uniform medium. Assuming an initial density distribution law as $\rho = \rho' r^{-\omega}$ where ρ' is the density at the centre and ω is a constant, analytical relations for the shock velocity, the pressure, the particle velocity, the sound velocity, the temperature and the change in entgropy of medium immediately behind the shock have been derived and their variations with propagation distance (r), constant (ω) and adiabatic index (γ) have been given through figures.

Finally, the dependence of these parameters on adiabatic index (γ), propagation distance (r) and constant (ω) have been discussed.

It is found that temperature and change in entropy of the medium increase as shock propagates. This agrees with earlier results [8, 13] as well as universal law of increase of entropy. For $\omega > \dfrac{4\gamma}{\left\{2+\sqrt{2\gamma(\gamma-1)}\right\}}$, the cange in entropy increase with ω and decrease with adiabatic index γ and the change in entropy is reverse for $\omega < \dfrac{4\gamma}{\left\{2+\sqrt{2\gamma(\gamma-1)}\right\}}$.

THEORY

The equations governing the spherically symmetrical flow enclosed by the shock front are :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0 \tag{1}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)\rho + \rho\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) = 0 \tag{2}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)\left(p\rho^{-\gamma}\right) = 0 \tag{3}$$

where, $u(r,t)$, $p(r,t)$ and $\rho(r,t)$ denote respectively the particle velocity, the pressure and density at a distance r from the origin at time t, and γ is the adiabatic index of gas.

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Let p_0 and ρ_0 denote the undisturbed value of pressure and density in front of shock and p_1, ρ_1 and u_1 be the values of respective quantities at any point immediately after passage of shock, the Rankine-Hugoniot conditions are :

$$p_1 = \rho_0 a_0^2 \left[\frac{2M^2}{\gamma+1} - \frac{\gamma-1}{\gamma(\gamma+1)} \right] \tag{4}$$

$$p_1 = \rho_0 \frac{(\gamma+1)M^2}{(\gamma-1)M^2+2}$$

$$u_1 = \frac{2a_0}{\gamma+1}\left(M - \frac{1}{M} \right)$$

$$a = a_0 \frac{\sqrt{\left[2\gamma M^2 - (\gamma-1)\right]\left[(\gamma-1)M^2+2\right]}}{(\gamma+1)M}$$

Where $M = U / a_0$ is Mach number, U is the shock velocity, a and a_0 are the sound velocities in disturbed and undisturbed medium respectively.

The temperature and change in entropy are given in MaGraw-Hill Encyclopedia of science and technology [3]

$$\frac{T}{T_0} = \frac{\left[2\gamma M^2 - (\gamma-1)\right]\left[2+(\gamma-1)M^2\right]}{(\gamma+1)^2 M^2} \tag{5a}$$

$$\frac{\Delta S}{R} = \frac{1}{\gamma-1} \ln \frac{\left[2\gamma M^2 - (\gamma-1)\right]\left[2+(\gamma-1)M^2\right]^{\gamma}}{(\gamma+1)^{\gamma+1} M^{2\gamma}} \tag{5b}$$

Where T and T_0 are the temperature of perturbed and unperturbed media respectively, ΔS is the change in entropy between perturbed and unperturbed media and R is the universal gas constant.

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For strong shock i.e. $M>>1$, conditions (4) and (5) reduce to

$$p=\frac{2\rho_0 U^2}{\gamma-1},\ \rho=\frac{\gamma+1}{\gamma-1}\rho_0,\ u=\frac{2U}{\gamma+1},\ a=\frac{\sqrt{2\gamma(\gamma-1)}}{\gamma+1}U \tag{6}$$

$$\frac{T}{T_0}=\frac{2\gamma(\gamma-1)}{(\gamma+1)^2}\left(\frac{U}{a_0}\right)^2 \tag{7a}$$

$$\frac{\Delta S}{R}=\frac{1}{\gamma-1}\ln\left\{\frac{2\gamma(\gamma-1)^\gamma}{(\gamma+1)^{\gamma+1}}\right\}\left(\frac{U}{a_0}\right)^2 \tag{7b}$$

For spherical diverging shocks, the characteristic form of the system of equations (1)-(3), i.e. the form in which each equation contains derivatives in only one direction in (r, t) plane, is

$$dp+\rho a du+\frac{2\rho a^2 u}{(u+a)}\frac{dr}{r}=0 \tag{8}$$

It is assumed that the medium in which shock propagates is non-uniform, whose initial density distribution is given by

$$\rho_0=\rho' r^{-\omega} \tag{9}$$

where ρ' is the density at the centre and ω is a constant.

Substituting conditions (6) into equation (8), we get

$$\frac{dU}{U}+\frac{1}{2+\sqrt{2\gamma/(\gamma-1)}}\left[-\omega+\frac{4\gamma}{2+\sqrt{2\gamma(\gamma-1)}}\right]\frac{dr}{r}=0 \tag{10}$$

On integrating equation (10), expression for shock velocity can be written as

$$U=Kr^{\left[\frac{1}{2+\sqrt{2\gamma/(\gamma-1)}}\left\{\omega-\frac{4\gamma}{2+\sqrt{2\gamma(\gamma-1)}}\right\}\right]} \tag{11}$$

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where K is the constant of integration.

The expression for non-dimensional shock velocity, there after called as shock strength can be written as

$$\frac{U}{a_0} = A\gamma^{-1/2} r^{\left[\frac{1}{2+\sqrt{2\gamma/(\gamma-1)}}\left\{\omega-\frac{4\gamma}{2+\sqrt{2\gamma(\gamma-1)}}\right\}\right]} \tag{12}$$

where $A = k\sqrt{\rho_0 / p_0}$

From equation (6), (7) and (12), the pressure, the particle velocity, the sound velocity, the temperature and the change in entropy are given by expressions (13)-(17) respectively.

$$\frac{p}{p_0} = \frac{2A^2}{(\gamma+1)} r^{\left[\frac{2}{2+\sqrt{2\gamma/(\gamma-1)}}\left\{\omega-\frac{4\gamma}{2+\sqrt{2\gamma(\gamma-1)}}\right\}\right]} \tag{13}$$

$$\frac{u}{a_0} = \frac{2A}{(\gamma+1)\sqrt{\gamma}} r^{\left[\frac{1}{2+\sqrt{2\gamma/(\gamma-1)}}\left\{\omega-\frac{4\gamma}{2+\sqrt{2\gamma(\gamma-1)}}\right\}\right]} \tag{14}$$

$$\frac{a}{a_0} = \frac{A\sqrt{2(\gamma-1)}}{(\gamma+1)} r^{\left[\frac{1}{2+\sqrt{2\gamma/(\gamma-1)}}\left\{\omega-\frac{4\gamma}{2+\sqrt{2\gamma(\gamma-1)}}\right\}\right]} \tag{15}$$

$$\frac{T}{T_0} = \frac{2A^2(\gamma-1)}{(\gamma+1)^2} r^{\left[\frac{2}{2+\sqrt{2\gamma/(\gamma-1)}}\left\{\omega-\frac{4\gamma}{2+\sqrt{2\gamma(\gamma-1)}}\right\}\right]} \tag{16}$$

$$\frac{\Delta S}{R} = \frac{1}{(\gamma-1)} In \frac{2A^2(\gamma-1)^{\gamma}}{(\gamma+1)^{\gamma+1}} r^{\left[\frac{2}{2+\sqrt{2\gamma/(\gamma-1)}}\left\{\omega-\frac{4\gamma}{2+\sqrt{2\gamma(\gamma-1)}}\right\}\right]} \tag{17}$$

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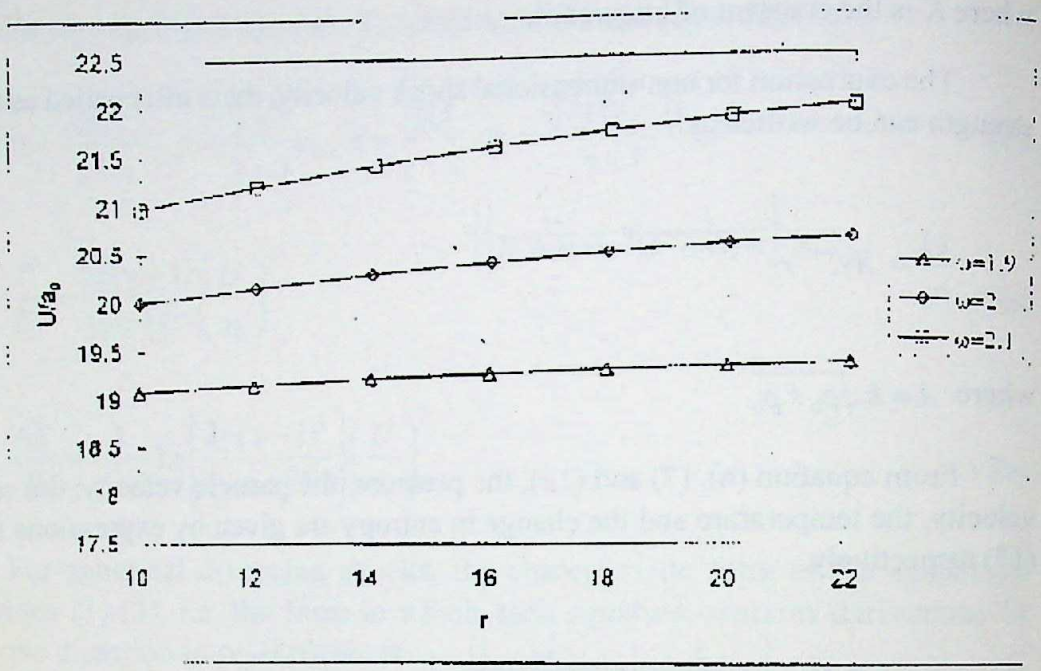

Figure 1 Variation of shock velocity (U/a_0) with propagation distance (r) for adiabatic index $\gamma=1.33$

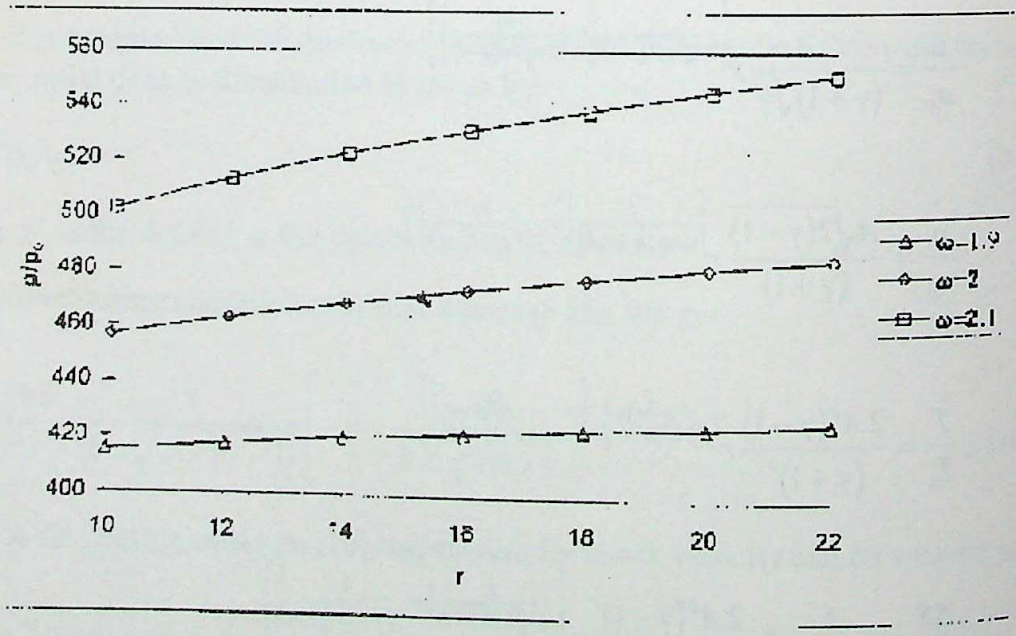

Figure 2 Variation of pressure (p/p_0) with propagation distance (r) for adiabatic index $\gamma=1.33$

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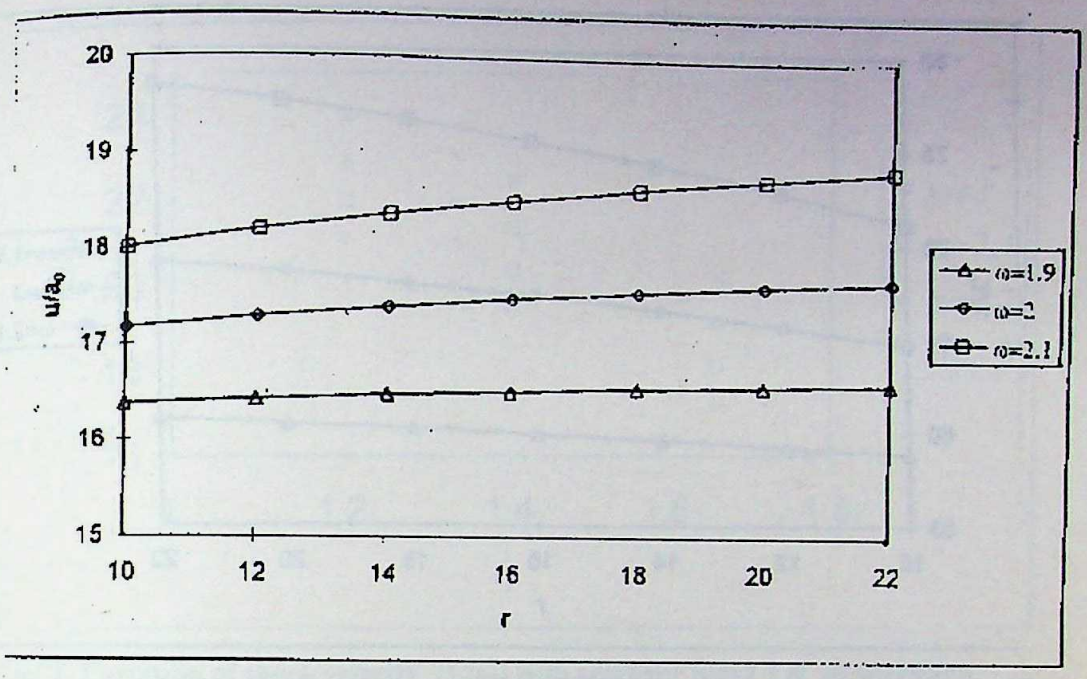

Figure 3 Variation of particle velocity (u/a_0) with propagation distance (r) for adiabatic index γ=1.33

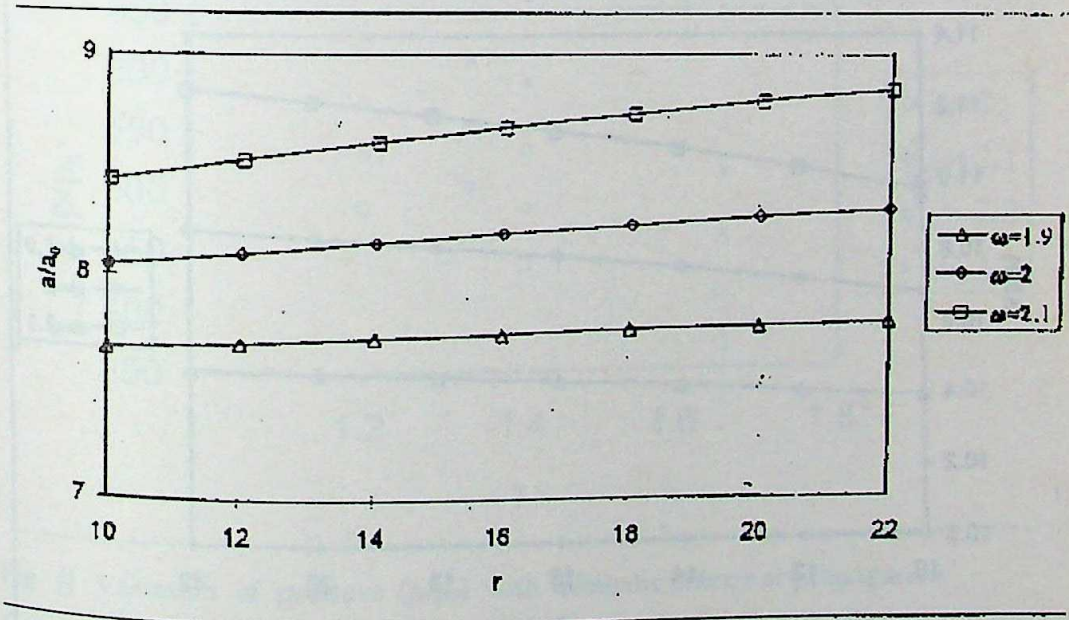

Figure 4 Variation of sound velocity (a/a_0) with propagation distance (r) for adiabatic index γ=1.33

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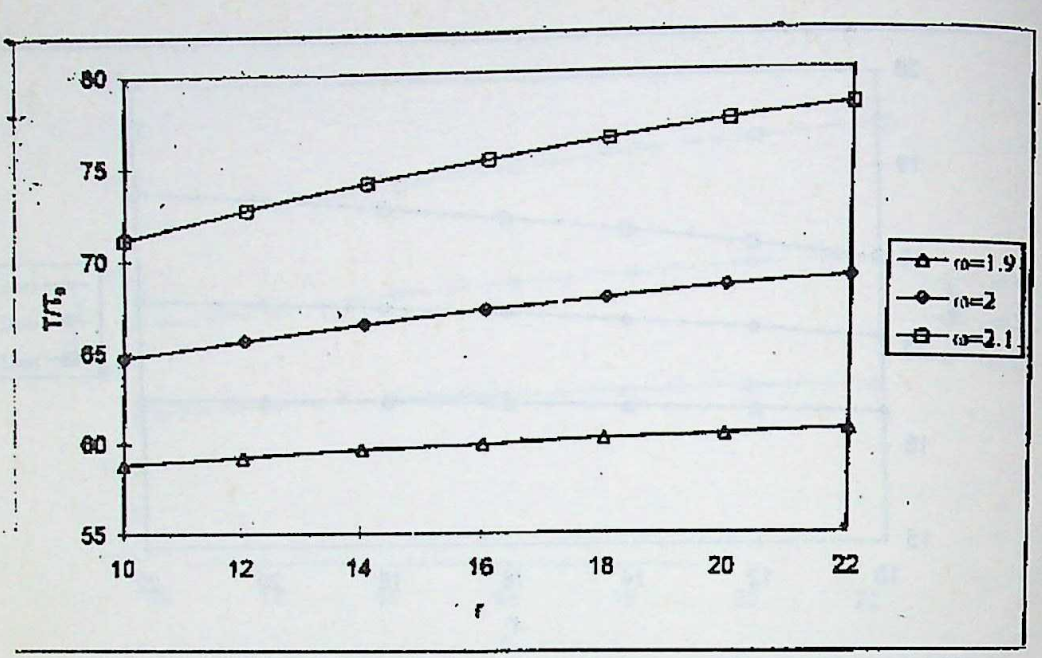

Fig 5 Variation of temperature (T/T_0) with propagation distance (r) for adiabatic index γ=1.33

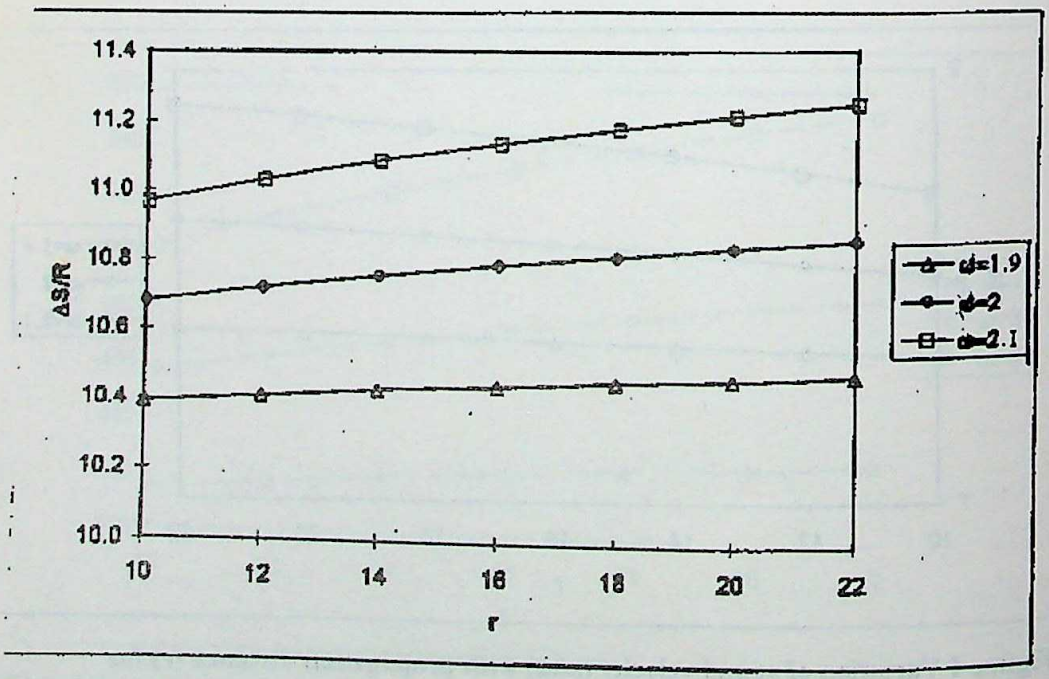

Fig 6 Variation of change in entropy ($\Delta S/R$) with propagation distance (r) for adiabatic index γ=1.33

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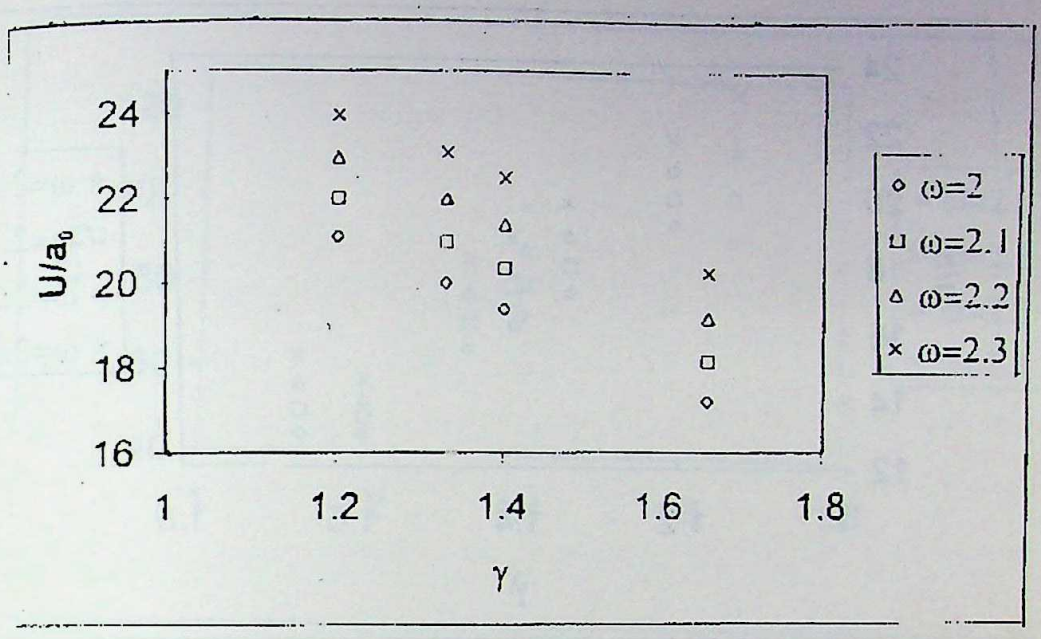

Fig 7 Variation of shock velocity (U/a_0) with adiabatic index γ at propagation distance r = 10

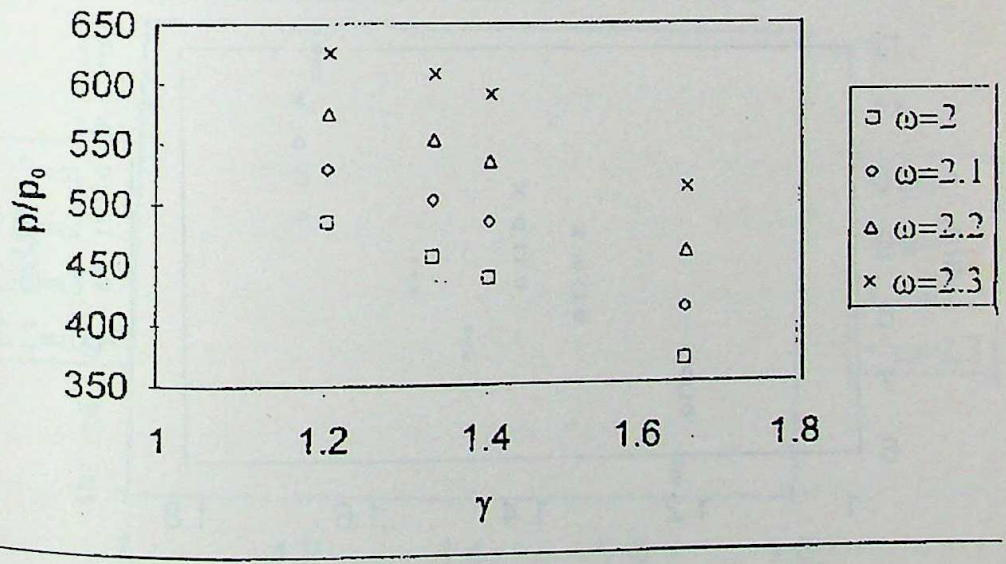

Fig 8 Variation of pressure (p/p_0) with adiabatic index γ at propagation distance r = 10

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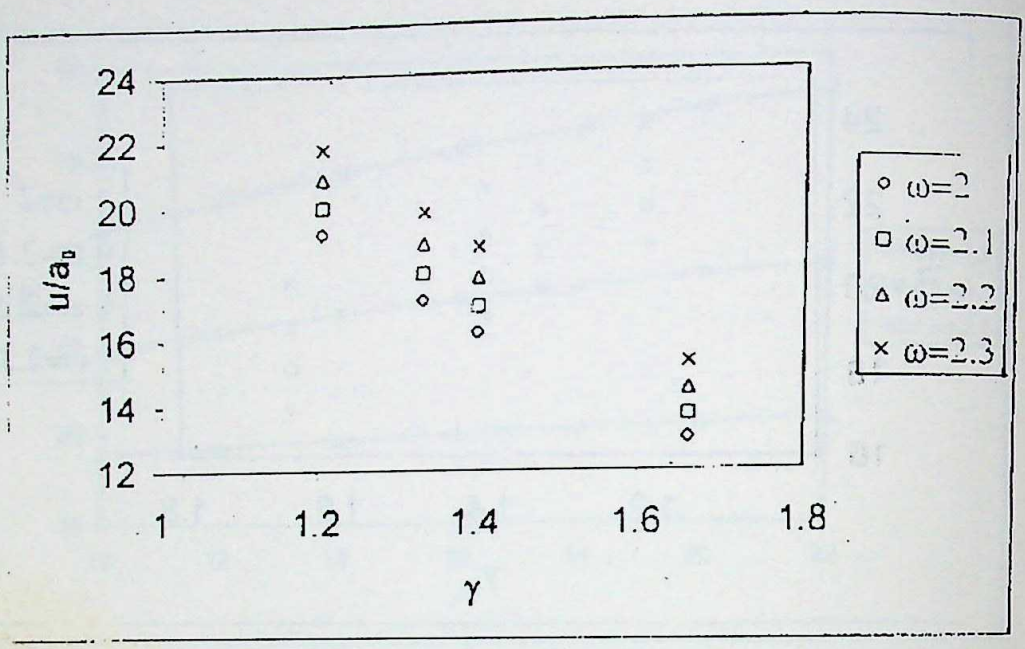

Fig 9 Variation of particle velocity (u/a_0) with adiabatic index γ at propagation distance $r = 10$

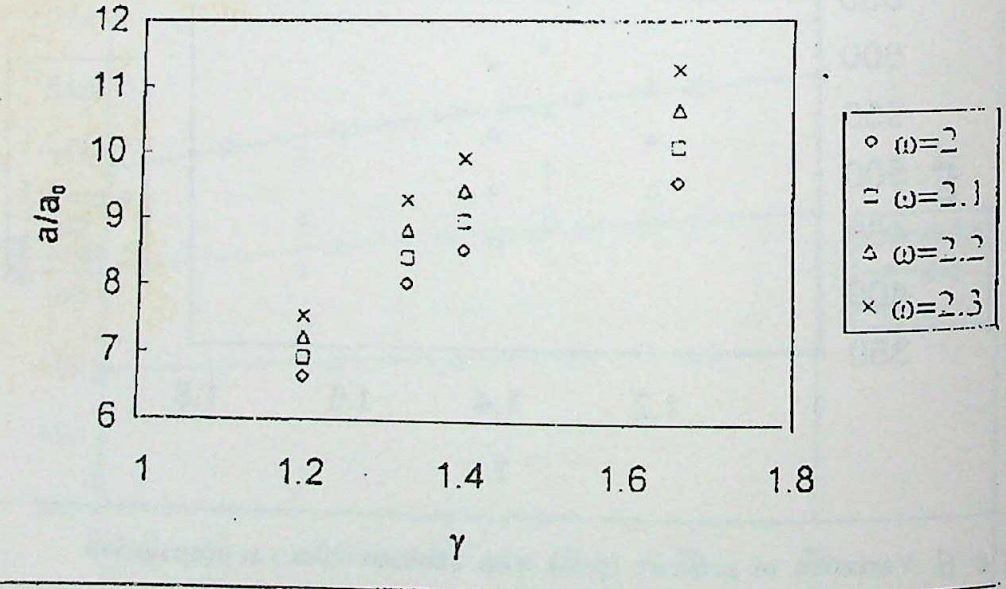

Fig 10 Variation of sound velocity (a/a_0) with adiabatic index γ at propagation distance $r = 10$

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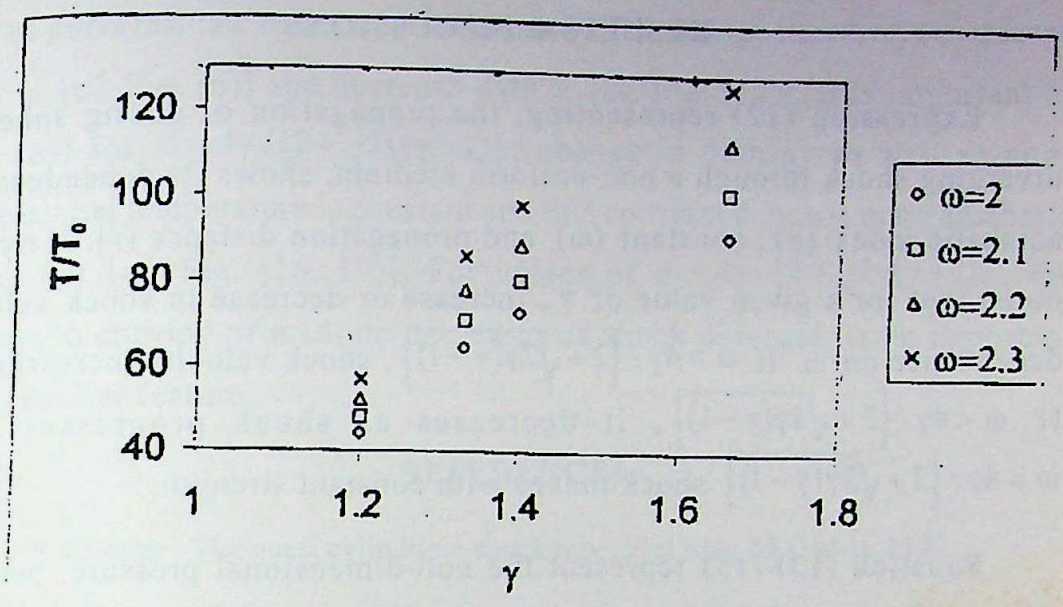

Fig 11 Variation of temperature (T/T_0) with adiabatic index γ at propagation distance r = 10

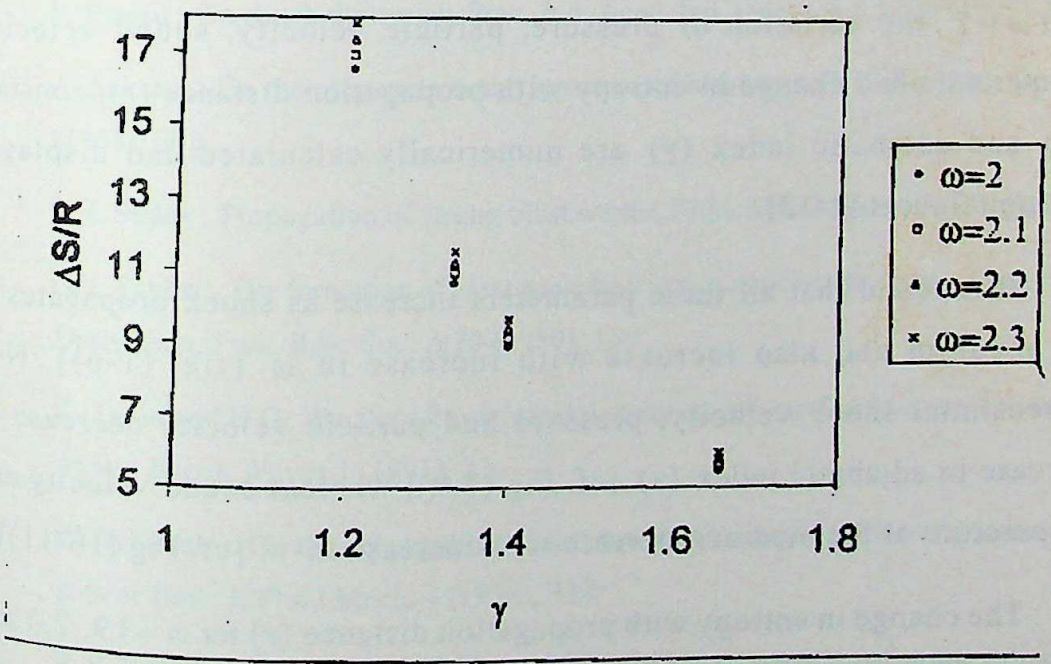

Fig 12 Variation of change in entropy (ΔS/R) with adiabatic index γ at propagation distance r = 10

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RESULTS & DISCUSSIONS

Expression (12) representing, the propagation of strong spherical diverging shock through a non-uniform medium, shows its dependence on adiabatic index (γ), constant (ω) and propagation distance (r). It may be noted that for a given value of γ, increase or decrease in shock velocity dependence on ω. If $\omega > 4\gamma / \left\{2+\sqrt{2\gamma(\gamma-1)}\right\}$, shock velocity increases and If $\omega < 4\gamma / \left\{2+\sqrt{2\gamma(\gamma-1)}\right\}$, it decreases as shock progresses. For $\omega = 4\gamma / \left\{2+\sqrt{2\gamma(\gamma-1)}\right\}$ shock moves with constant strength.

Equation (13)-(15) represent the non-dimensional pressure, particle velocity and sound velocity immediately behind the shock of the medium as strong spherical diverging shock propagates, respectively.

Taking the initial shock strength $U/a_0 = 20$ at $r = 10$, for $\gamma = 1.33$ and $\omega = 2$, the variation of pressure, particle velocity, sound velocity, temperature and change in entropy with propagation distance (r), constant (ω) and adiabatic index (γ) are numerically calculated and displayed through figures (1-12).

It is found that all these parameters increase as shock propagates in the medium and also increase with increase in ω [fig. (1-6)]. Non dimensional shock velocity, pressure and particle velocity decrease as increase in adiabatic index (γ) [cf. Fig (7-9)] whereas sound velocity and temperature of the medium increase with increase in γ. [cf. Fig (10-11)].

The change in entropy with propagation distance (r) for $\omega = 1.9$, 2.0 and 2.1 has been shown in fig. (6). It is seen that change in entropy of medium increases as shock propagates. This agrees with earlier results [8, 13] as

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well as universal law of increase of entropy. The change in entropy increases with ω [cf. Fig. (6)] and decrease with γ [cf. Fig. (12)]. It is important to note that for $\omega = 4\gamma / \left\{2 + \sqrt{2\gamma(\gamma-1)}\right\}$, change in entropy as well as non-dimensional temperature is constant and this constant depends upon adiabatic index (γ) [cf. Fig. (16, 17)]. For values of $\omega < 4\gamma / \left\{2 + \sqrt{2\gamma(\gamma-1)}\right\}$, the change in entropy of medium decreases as shock diverges, is an important and peculiar feature.

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HOPF-BIFURCATION AND STABILITY OF A 2-PREDATORS, 1-PREY SYSTEM

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ABSTRACT

This paper considers a system modelling the interaction of two competing predatory populations with density-dependent death rats, living exclusively on a common prey. The existence of interior positive equilibrium (coexistence) is proven and sufficient conditions for its asymptotic stability are given. Furthermore, a derivation of an easily verifiable conditions for the existence of a hopf-bifurcating one-parameter family periodic solutions emanating from the interior positive equilibrium of a general of species predator-prey system are given. The stability of the bifurcating periodic 3-solutions is shown, through an example, to be an open question.

Key words: 3-species predator-prey system, hopf-bifurcation, equilibrium, periodic solution, stability.

Mathematics Subject Classification (2000) : 92 B05

INTRODUCTION

For a predator-prey system (see [1]) it is well known that the existence and the stability of a positive equilibrium is essential to investigate the existence of non-trivial periodic solutions. This question becomes even more interesting and important when two or more predatory populations compete for a single prey population. In this work, we have considered a system of three O.D.E.,s modelling the interaction of two predatory populations living exclusively on a common prey population. For both predatory populations, we have incorporated density-dependent death rates as opposed to constant death rates as considered in [3] and others, a situation that does not appear to be realistic. For a system with constant death rates, there may be no positive equilibrium existing and hence constant coexistence of the species is excluded. However, it is posibble that these competing predators together with their sole food source may coexist in a periodic fashion. In fact this does happen and the three species coexist in the form of state peiodicity see [10, 11].

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In the case, however of a system with density-dependent death rates, we have proved the existence of the interior positive equilibrium, its asymptotic stability, and existence of bifurcating periodic solutions emanating from this equilibrium point.

The organization of this paper is as follows.

In section 2, we give the model and discuss the asymptotic stability of equilibria in the co-ordinate planes.

Section 3 deals with the existence and asymptotic stability of the interior positive equilibrium. In section 4, we derive conditions for the existence of Hopf-bifurcating periodic orbits emanating from the interior positive equilibrium of a general 3-species system. We also show by an example that the question of stability of the bifurcating periodic orbit is inconclusive.

THE MODEL

Hypothesis on the Model

A system modelling the interspecific interactions of two competing predator populations and living exclusively on a common prey is of the form :

$$\begin{aligned} \dot{x} &= xf(x) - yp(x) - zq(x) \\ \dot{y} &= y\left[-g(y) + c_1 p(x)\right] \\ \dot{z} &= z\left[-h(z) + d_1 q(x)\right], \\ x(0) &= x_0 > 0,\ y(0) = y_0 > 0,\ z(0) = z_0 > 0. \end{aligned} \tag{2.1.1}$$

where

$\left(\cdot = \frac{d}{dt}\right)$, c_1, d_1 are positive constants known as food (biomass) conversion rates.

We make the following assumptions, which are consistent with models of predator-prey system.

(H1) : $f(x)$; specific growth rate of prey x, $f(0) > 0$, $f'(x) \leq 0$ for $x \geq 0$, there is $k > 0$ (the environmental carrying capacity) such that $f(x) > 0$ on $0 \leq x < k$, $f(k) = 0$, and $f(x) < 0$ on $x > k$.

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(H2): $p(x)$; the functional response of predator y with respect to prey x, and $p(0)=0$, $p'(x)>0$ for $x>0$.

(H3): $q(x)$; the functional response of predator z with respect to prey x, and $q(0)=0$, $q'(x)>0$ for $x>0$.

(H4): $q(x)=\alpha p(x)$, α is a positive constant.

(H5): $g(y)$; density-dependent death rate of predator y, $g(0)>0$, and $g'(y)>0$ for $y>0$.

(H6): $h(z)$; density-dependent death rate of predator z, $h(0)>0$, and $h'(z)>0$ for $z>0$.

Asymptotic Stability of Equilibria in Co-ordinate Planes

Clearly the system (2.1.1.) has an equilibrium at $E_0(0,0,0)$ which is hyperbolic and unstable along the x-axis.

By the assumption (H1), $E_1(k,0,0)$ [a predator-fee state] is also an equilibrium point. The eigenvalues of the variational matrix $V(E_1)$ of the system (2.1.1.) about $E_1(k,0,0)$ are :

$$\lambda = kf'(k)<0,\ \lambda_2 = -g(0)+c_1p(k),\ \lambda_3 = -h(0)+d_1q(k) \qquad (2.2.1)$$

Remark 2.2.1. From (2.2.1.) it follows that $E_1(k,0,0)$ is asymptotically stable along the x-axis. This implies that the prey population remains in a neighboured of k. For existence of $\hat{E}(\hat{x},\hat{y},0)$ and $\tilde{E}(\tilde{x},0,\tilde{z})$ as equilibria, it is necessary as well as sufficient that :

$$-g(0)+c_1p(k)>0 \text{ and } -h(0)+d_1q(k)>0 \qquad (2.2.2.)$$

as it implies increase of predator populations $y(t)$ and $z(t)$ respectively, see [4]. Thus we assume that each of the predators y and z can survive on the prey x in the absence of its rival, that is, there exist equilibria $\hat{E}(\hat{x},\hat{y},0)$ and $\tilde{E}(\tilde{x},0,\tilde{z})$ satisfying :

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$$\left.\begin{aligned}\hat{x}f(\hat{x})-\hat{y}f(\hat{x})&=0\\ -g(\hat{y})+c_1p(\hat{x})&=0\end{aligned}\right\} \tag{2.2.3.}$$

and

$$\left.\begin{aligned}\tilde{x}f(\tilde{x})-\tilde{z}q(\tilde{x})&=0\\ -h(\tilde{z})+d_1q(\tilde{x})&=0\end{aligned}\right\} \tag{2.2.4.}$$

respectively.

Lemma 2.2.1. If $\dfrac{d}{dx}\left(\dfrac{xf(x)}{p(x)}\right)_{x=\hat{x}}<0$ and $h(0)-d_1q(\hat{x})>0$, then the quilibrium $\hat{E}(\hat{x},\hat{y},0)$ is asymptotically stable.

Proof : The entries (a_{ij}) of the variational matrix $V(\hat{E})$ of the system (2.1.1.) about $\hat{E}(\hat{x},\hat{y},0)$ (where simplification is done using (2.2.3.)) are :

$$\left.\begin{aligned}&a_{11}=p(\hat{x})\frac{d}{dx}\left(\frac{xf(x)}{p(x)}\right)_{x=\hat{x}}<0,\quad a_{12}=-p(\hat{x})<0,\quad a_{13}=-q(\hat{x})<0\\ &a_{21}=c_1\hat{y}p(\hat{x})>0,\qquad a_{22}=-\hat{y}g'(\hat{y})<0,\quad a_{23}=0\\ &a_{31}=0,\qquad\qquad a_{32}=0,\qquad\qquad a_{33}=-\left(h(0)-d_1q(\hat{x})\right)<0\end{aligned}\right\} \tag{2.2.5.}$$

The characteristic equation of $V(\hat{E})$ is

$$\lambda^3+p_1\lambda^2+p_2\lambda+p_3=0, \tag{2.2.6.}$$

where

$$\left.\begin{aligned}p_1&=-(a_{11}+a_{22}+a_{23})\\ p_2&=(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33})-(a_{12}a_{21}+a_{13}a_{31}+a_{23}a_{32})\\ p_3&=(a_{11}a_{23}a_{32}+a_{22}a_{31}a_{13}+a_{33}a_{12}a_{21})-(a_{11}a_{22}a_{23}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32})\end{aligned}\right\} \tag{2.2.7.}$$

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From (2.2.5.) and (2.2.7.), we have

$$p_1 > 0,\ p_3 > 0,\ \text{and}\ p_1 p_2 - p_3 > 0.$$

Thus by Routh-Hurwitz criterion, $\hat{E}(\hat{x},\hat{y},0)$ is asymptotically stable.

Lemma 2.2.2. If $\left.\frac{d}{dx}\left(\frac{xf(x)}{p(x)}\right)\right|_{x=\tilde{x}} < 0$ and $g(0) - c_1 p(\tilde{x}) > 0$, then the equilibrium $\tilde{E}(\tilde{x},0,\tilde{z})$ is asymptotically stable.

Proof : The proof is similar to that of lemma 2.2.1.

Remark 2.2.2. There is no equilibrium in the y-z plane, for if we suppose that there does exist an equilibrium $E^*(0,y^*,z^*)$ in the y-z plane which is given by $g(y^*) = 0$ and $h(z^*) = 0$, then this contradicts assumptions (H5) and (H6). Moreover, it is, indeed, consistent with the intuitive ecological observation that predators can not coexist while their sole food source is extinct.

EXISTENCE OF INTERIOR POSITIVE EQUILIBRIUM AND ITS STABILITY

For the system (2.2.1.) with $g(y)$ and $h(z)$ as contant functions, the question whether or not it is possible to have all three species x, y and z coexisting is a quite significant and interesting question both from mathematical and biological points of view.

In general, no steady state $x \neq 0,\ y \neq 0,\ z \neq 0$ exists in this case.

This is because the algebraic equations (say with Holling type II functional responses)

$$xf(x) - yp(x) - zq(x) = 0$$

$$-s + c_1 p(x) = 0$$

$$-d + d_1 q(x) = 0$$

do not have a simultaneous solution. In the degenerate case, $x_1 = x_2 < k$, there exists a line segment of steady states joining $\hat{E}(\hat{x},\hat{y},0)$ and $\tilde{E}(\tilde{x},0,\tilde{z})$. This implies, in general,

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that the competing predators would exclude coexistence in a constant steady state situation. Since no constant equilibrium exists, this could occur only through a stable time varying solutions. In fact, this does happen and the ghree species can coexist in the form of a stable periodic trajectory see [10, 11].

However, when both $g(y)$ and $h(z)$ are not constant functions, there does exist an interior positive equilibrium as we show now.

Existence of Positive Equilibrium

For the existence of positive equilibrium, we give an indirect proof. If such a positive equilibrium $E^*(x^*, y^*, z^*)$ of the system (2.1.1.) exists then it is given by :

$$x^* f(x^*) - y^* p(x^*) - z^* q(x^*) = 0$$

$$-g(y^*) + c_1 p(x^*) = 0 \qquad (3.1.1.)$$

$$-h(z^*) + d_1 q(x^*) = 0$$

As shown in [2, 4] the notions of global asymptotic stability and persistence of the system are connected in the sense that the criteria for persistence of the system (i.e. the persistence of top predator populations in a simple food chain. Indeed, in our system (2.1.1.) each predator acts as a top predator, consequently imply the existence of a positive equilibrium.

Thus, for the existence of positive equilibrium we should have the following conditions:

(1) Range $(g) = c_1$ Range (p), Range $(h) = d_1$ Range (g), and g^{-1} and h^{-1} both exist.

(2) $\frac{xf(x)}{p(x)}$ and $\frac{xf(x)}{q(x)}$ are both strictly decreasing functions in the positive cone. This for the equilibrium to be globally asymptotically stable (see Remark 3.2.1.).

(3) $-g(0) + c_1 p(\tilde{x}) > 0$ and $-h(0) + d_1 q(\hat{x}) > 0$ where $\hat{E}(\hat{x}, \hat{y}, 0)$ and $\tilde{E}(\tilde{x}, 0, \tilde{z})$ are equilibria in the co-ordinate planes (see section 2.2.). These two conditions are for persistence of the system (2.1.1) (see [2,3,4]).

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Remark 3.1.1.

The condition that $\frac{xf(x)}{p(x)}\left(\frac{xf(x)}{q(x)}=\frac{xf(x)}{\alpha p(x)}\text{ when (H4) holds}\right)$ is strictly decreasing also implies global asymptotic stability of the equilibria $\hat{E}(\hat{x},\hat{y},0)$ and $\tilde{E}(\tilde{x},0,\tilde{z})$, that is non-existence of limit cycles in the x-y plane and x-z plane.

Thus conditions (1), (2), (3) imply the existence of positive equilibrium.

We illustrate this by two examples.

Example 3.1.1.

Consider the system (2.1.1.) with logistic growth of the prey population and Holling type II functional responses.

$$\dot{x}=x(1-x)-y\frac{2x}{1+x}-z\frac{4x}{1+x}$$

$$\dot{y}=y\left[-(1+y)+\frac{33}{16}\frac{2x}{1+x}\right] \qquad (3.1.2.)$$

$$\dot{z}=z\left[-(1+z)+\frac{11}{9}\frac{4x}{1+x}\right]$$

Condition (1) : is trivially satisfied.

Condition (2) :

$$\frac{d}{dx}\left(\frac{xf(x)}{p(x)}\right)=-x<0, \qquad 0<x^{*}\le x<k$$

$$\frac{d}{dx}\left(\frac{xf(x)}{q(x)}\right)=-\frac{1}{2}<0, \qquad 0<x^{*}\le x<k$$

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for condition (3) :

$$\hat{E}(\hat{x},\hat{y},0)=\hat{E}\left(\frac{1}{2},\frac{3}{8},0\right)$$

$$\tilde{E}(\tilde{x},0,\tilde{z})=\tilde{E}\left(\frac{1}{3},0,\frac{8}{36}\right)$$

$$-g(0)+c_1p(\tilde{x})=-1+\frac{33}{16}\left(p\left(\frac{1}{3}\right)\right)=\frac{1}{32}>0$$

$$-h(0)+d_1q(\hat{x})=-1+\frac{11}{9}q\left(\frac{1}{2}\right)=\frac{17}{27}>0,$$

and we have $E^*(x^*,y^*,z^*)=E^*(.329,.02,.21)$ a positive equilibrium for (3.12.).

Example 3.1.2.

Consider the system :

$$\begin{aligned}\dot{x}&=x(1-x)-yx-2xz\\ \dot{y}&=y[-2(2+y)+3x]\\ \dot{z}&=z[-(2+z)+4x]\end{aligned}\qquad(3.1.3.)$$

condition (2) is satisfied.

$$\hat{E}(\hat{x},y,0)=\hat{E}\left(\frac{3}{4},\frac{1}{4},0\right)$$

$$\tilde{E}(\tilde{x},0,\tilde{z})=\hat{E}\left(\frac{5}{9},0,\frac{5}{9}\right)$$

$-h(0)+d_1q(\hat{x})=1>0$, but

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$-g(0)+c_1p(\tilde{x})=-2+3.\frac{5}{9}=-\frac{1}{3}<0$, thus (3) is not satisfied implying that (3.1.3.) may have no positive equilibrium.

No solving

$$\left.\begin{aligned}1-x-y-2z&=0\\2+y&=3x\\2+z&=4x\end{aligned}\right\}\Rightarrow x=\frac{7}{12},y=-\frac{1}{4},z=\frac{1}{3}$$

which in not a positive equilibrium of (3.1.3.).

3.2. Asymptotic Stability of $E^*(x^*,y^*,z^*)$.

The asymptotic stability of the equilibrium $E^*(x^*,y^*,z^*)$ for (2.1.1.) is proved under the assumption (H4), that is

$$q(x)=\alpha p(x). \tag{3.2.1.}$$

where α is taken as a positive parameter which will be of significance for the Hopf-bifurcation in section 4.

Theorem 3.2.1.

If $\frac{d}{dx}\left(\frac{xf(x)}{p(x)}\right)_{x=x^*}<0$, then the equilibrium $E^*(x^*,y^*,z^*)$ is asymptotically stable.

Proof :

The entries (a_{ij}) of the variational matrix $V(E^*)$ of the system (2.1.1.) about $E^*(x^*,y^*,z^*)$ (as simplification is done using (3.1.1.)) are

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$$\left.\begin{aligned} a_{11} &= p(x^*)\frac{d}{dx}\left(\frac{xf(x)}{p(x)}\right)_{x=x^*} < 0, & a_{12} &= -p(x^*) < 0, & a_{13} &= -\alpha p(x^*) < 0 \\ a_{21} &= y^* c_1 p(x^*) > 0, & a_{22} &= -y^* g'(y^*) < 0, & a_{23} &= 0 \\ a_{31} &= \alpha d_1 z^* p(x^*) > 0, & a_{32} &= 0, & a_{33} &= -z^* h'(z^*) < 0 \end{aligned}\right\} \quad (3.2.2.)$$

From (2.2.6.) and (3.2.2.) we have $p_1 > 0, p_3 > 0$ and $p_1 p_2 - p_3 > 0$.

Thus by the Routh-Hurwitz criterion, $E^*(x^*, y^*, z^*)$ is asymptotically stable.

Remark 3.2.1.

For the global asymptotic stability of $E^*(x^*, y^*, z^*)$, we define a Lyapunov function on $\{(x,y,z): 0 < x^* \le x < k, 0 < y^* \le y < \beta, 0 < z^* \le z < \gamma, \beta > 0, \gamma > 0\}$ as

$$V(x,y,z) = \int_{x^*}^{x} (c_1 + d_1)\left(1 - \frac{p(x^*)}{p(w)}\right)dw + \int_{y^*}^{y}\left(\frac{w - y^*}{w}\right)dw + \int_{z^*}^{z}\left(\frac{w - z^*}{w}\right)dw \quad (3.2.3.)$$

It is straightforward to show that

$\dot{V}(x,y,z) < 0$ provided $\dfrac{xf(x)}{p(x)}$ is a strictly decreasing function on $0 < x^* \le x < k$.

HOPF-BIFURCATION IN 3-SPECIES SYSTEM

Hopf-bifurcation is one of the most common ways through which a constant steady state loses its stability giving rise to a solution (or a family of solutions) of a periodic nature.

The technique of Hopf-bifurcation relies heavily on the effects induced by the parameter incorporated in the system on the stability of the steady state of the system. Thus, Hopf-bifurcation is characterized by a stability change of the steady state accompanied by the creation of limit cycles.

The Hopf-bifurcation theorem is a very powerful general tool in establishing the existence of a one-parameter family of periodic solutions and in a neighborhood

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of the bifurcation point, it also gives their periods. However, it does not immediately provides any information about the stability of these bifurcating periodic solutions.

Various successful attempts have been made to ascertain the orbital stability of the bifurcating periodic solution such as stability formula as in [6], the procedure applied in [5], center manifold theory and the constant "a" value technique in [12], and the method of Negrini-Salvadori [7]. Nevertheless, we show by an example that the stability of the bifurcating periodic solution seems to be an open question.

Conditions for Existence of Bifurcating Periodic Solutions

In this section we obtain conditions in terms of p_1, p_2, p_3 as defined in (2.2.7.) for the existence of bifurcating periodic solution emanating from the positive equilibrium $E^*(x^*, y^*, z^*)$ of the general 3-species system

$$\dot{x}_i = x_i f_i(x_i, y_i, z_i, \alpha), \quad i = 1,2,3 \tag{4.1.1.}$$

α being a parameter.

We shall be seeking conditions on the characteristic equation of the variational matrix $V(E^*)$, i.e.

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 = 0 \tag{4.1.2.}$$

to have 2 purely imaginary roots and a criterion so as the transversality condition holds.

Suppose (4.1.2.) has one real root λ_1 and two complex conjugate roots :

$$\lambda_2 = \lambda_r + i\lambda_s, \quad \lambda_3 = \lambda_r - i\lambda_s.$$

Thus

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

$$\lambda^3 - \lambda^2(2\lambda_r + \lambda_1) + \lambda\left(|\lambda_2|^2 + 2\lambda_r\lambda_1\right) - |\lambda_2|^2\lambda_1.$$

By comparing coefficients of like terms, we have

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$$\left.\begin{aligned} -p_1 &= 2\lambda_r + \lambda_1 \\ p_2 &= |\lambda_2|^2 + 2\lambda_r\lambda_1 \\ p_3 &= -|\lambda_2|^2\lambda_1 \end{aligned}\right\} \qquad (4.1.3.)$$

Let us consider α, as given in (3.2.1.), to be the Hopf-bifurcation parameter which induces (by tunning it about a critical value) the change in stability status of the equilibrium $E^*(x^*, y^*, z^*)$.

Now (4.1.2.) has to have two purely imaginary roots if $\lambda_r = 0$.

Thus from (4.1.3.) we have

$$p_1 p_2 - p_3 = 0 \qquad (4.1.4.)$$

Equation (4.1.4.) gives the critical value of α, call it α_c, such that $p_1 p_2 - p_3 = 0$.

Thus α_c satisfies $\bar{p}_1 \bar{p}_2 - \bar{p}_3 = 0$,

where $\bar{p}_i (i = 1,2,3)$ denotes the value of p when α is replaced by α_c.

Thus the two purely imaginary roots are $\pm iw$, where $w = \sqrt{p_2}$ whit $p_2 > 0$. This is because if λ_1 is a real root of (4.1.2.) then the other two are give by

$$\lambda_{2,3} = \left[-(\lambda_1 + p_1) \pm \sqrt{(\lambda_1 + p_1)^2 - 4(\lambda_1^2 + \lambda_1 p_1 + p_2)}\right] \Big/ 2$$

If $\lambda_{2,3}$ are purely imaginary then $\lambda_1 = -p_1$ and it implies that $\lambda_{2,3} = \pm i\sqrt{p_2}$ with $p_2 > 0$.

Next to find a transversality condition in terms of p_1, p_2, p_3 such that $\frac{d}{dx}(\lambda_r(\alpha))_{\alpha=\alpha_c} \neq 0$

Now eliminating λ_1 from (4.1.3.) will result in

$$8\lambda_r^3 + 8\lambda_r^2 p_1 + 2\lambda_r[p_2^2 + p_1 p_3] + p_1 p_2 - p_3 = 0 \qquad (4.1.5.)$$

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Differentiating (4.1.5.) with respect to α, then putting $\alpha=\alpha_c$, and observing that $\lambda_r(\alpha_c)=0,\ (p_1p_2-p_3)_{\alpha=\alpha_c}=0,$ we get

$$2\left[p_1^2+p_2\right]_{\alpha=\alpha_c}\frac{d}{d\alpha}\left(\lambda_1(\alpha)\right)_{\alpha=\alpha_c}=-\frac{d}{d\alpha}(p_1p_2-p_3)_{\alpha=\alpha_c}$$

$$\therefore\ \frac{d}{d\alpha}\left(\lambda_r(\alpha)\right)_{\alpha=\alpha_c}=-\frac{1}{2\left[p_1^2+p_2\right]}\frac{d}{d\alpha}(p_1p_2-p_3)_{\alpha=\alpha_c} \qquad (4.1.6.)$$

Hence

$$p_1^2+p_2>0,\ \frac{d}{d\alpha}\left(\lambda_r(\alpha)\right)>0(<0)\ \text{if}\ \frac{d}{d\alpha}(p_1p_2-p_3)<0(<0) \qquad (4.1.7.)$$

Remark 4.1.1.

Since for $d=\frac{d}{d\alpha}\left(\lambda_r(\alpha)\right)_{\alpha=\alpha_c}>0,$ the eigenvalues cross from left half-plane to the right-half plane as α increases, it follows that the equilibrium is asymptotically stable for $\alpha<\alpha_c$ and unstable for $\alpha>\alpha_c$.

For $d<0$, the eigenvalues cross from the right half-plane to the left half-plane as α increases, it follows that the equilibrium point is unstable for $\alpha<\alpha_c$ and asymptotically stable for $\alpha>\alpha_c$.

This means that for $d>0\left(i.e.\ \frac{d}{d\alpha}(p_1p_2-p_3)<0\right)$, if $F(\alpha)=p_1p_2-p_3$, then $F(\alpha)$ is decreasing at $\alpha=\alpha_c$, which implies that $p_1p_2-p_3<0$ for $\alpha>\alpha_c$, $\alpha-\alpha_c$ to be small.

Thus with $p_1>0$ and $p_3>0$, the asymptotic stability of $E^*\left(x^*,y^*,z^*\right)$ depends upon $p_1p_2-p_3$. Hence, for $\alpha<\alpha_c, p_1p_2-p_3>0$ we have $E^*\left(x^*,y^*,z^*\right)$ asymptotically stable, while for $\alpha>\alpha_c$, $\alpha-\alpha_c$ is small, $p_1p_2-p_3<0$, we have $E^*\left(x^*,y^*,z^*\right)$

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unstable, whereas for $\alpha = \alpha_c$, $p_1 p_2 - p_3 = 0$ there is a Hopf-bifurcation.

Thus we have proved the following theorem :

Theorem 4.1.1.

The interior equilibrium point in the positive cone of the system (2.1.1.) is asymptotically stable for $\alpha < \alpha_c$, unstable for $\alpha > \alpha_c$, and there is a Hopf-bifurcation at $\alpha = \alpha_c$, where α_c is given by (4.1.4.).

However; for $\frac{d}{d\alpha}(p_1 p_2 - p_3)_{\alpha=\alpha_c} > 0 \left(i.e. \frac{d}{d\alpha}(\lambda_r(\alpha))_{\alpha=\alpha_c} < 0 \right)$, then theorem 4.1.1. reads as :

Theorem 4.1.2.

The interior equilibrium point in the positive cone of the system (2.1.1.) is asymptotically stable for $\alpha > \alpha_c$, unstable for $\alpha < \alpha_c$, and there is a Hopf-bifurcation at $\alpha = \alpha_c$, where α_c is given by (4.1.4.)

We illustrate the two theorem by the following examples.

Example 4.1.1.

Consider the sysem (The classical Lorenz equations), see [6]

$$\dot{x} = -\sigma x + \sigma y$$

$$\dot{y} = -xz + ry - y \qquad (4.1.8.)$$

$$\dot{z} = xy + -bz$$

where b and σ are positive constants and r to be taken as a Hopf-bifurcation parameter. $E^*(x^*, y^*, z^*) = E^*\left(\sqrt{b(r-1)}, \sqrt{b(r-1)}, (r-1)\right)$, whit $r > 1$.

Moreover, $p_1 = 1 + b + \sigma$, $p_2 = (r + \sigma)b$, $p_{3=} = 2\sigma b(r-1)$.

$p_1 p_2 - p_3 = 0$ implies that the critical value of the parameter r is

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$$r_c = \frac{\sigma(\sigma+b+3)}{\sigma-(b+1)},\quad \frac{d}{dr}(p_1p_2-p_3)_{r=r_c} = b(1+b-\sigma)<0 \text{ provided that } \sigma>1+b.$$

Thus Hopf-bifurcation will occur at $r=r_c$ provided that $\sigma>1+b$ and this condition is the same as in [6].

Example 4.1.2.

Consider the following predator-prey system, see [8] :

$$\dot{x} = x\left(1-\frac{\alpha k}{\in}y-z\right)$$

$$\dot{y} = \frac{y}{\in}(-y+\beta kz) \qquad (4.1.9.)$$

$$\dot{z} = \frac{a}{\in}(x-z)$$

which is a predator-prey model with time lag, where $\alpha, k, \in, \gamma, \beta$ are positive constants and $a>0$ be taken as the Hopf-bifurcation parameter. The positive equilibrium point is $E^*(x^*,y^*,z^*) = E^*\left(\frac{\gamma}{\beta k}, \frac{\in}{\alpha k}\left(1-\frac{\gamma}{\beta k}\right), \frac{\gamma}{\beta k}\right)$ with $\beta k-\gamma>0$.

Thus we get

$$p_1 = \frac{a}{\in},\ p_2 = \frac{a\gamma}{\in\beta k},\ p_3 = \frac{\gamma a}{\in^2\beta k}(\beta k-\gamma), \text{ hence}$$

$$p_1p_2-p_3 = \frac{a\gamma}{\in^2\beta k}(a-\beta k+\gamma)=0 \text{ implies } a_c=\beta k-\gamma>0$$

(a_c is the parameter critical value at which Hopf-bifurcation occurs).

For the transversality condition we have :

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$\frac{d}{da}(p_1p_2 - p_3)_{a=a_c} = \frac{\gamma}{\epsilon^2 \beta k}(\beta k - \gamma) > 0$ which implies $\frac{d}{d\alpha}(\lambda_r(a))_{a=a_c} < 0$ and hence the equilibrium point $E^*(x^*, y^*, z^*)$ is asymptotically stable for $a > a_c$, unstable for $a < a_c$ and Hopf-bifurcation occurs at $a = a_c$.

Stability of Bifurcating Periodic Orbit

In this section we illustrate by an example that the stability of the bifurcating periodic solution seems to be an open question.

Example 4.2.1.

Consider a 3-species predator-prey model in which two species (x and y say) at the bottom are feeding on each other and at the same time are fed upon by the third species (z) acting as the top predator of the system. This system can be represented as :

$$\left.\begin{aligned} \dot{x} &= x[1 - x + \alpha y - \beta z] \\ \dot{y} &= y[1 - y + \alpha x - \beta z] \\ \dot{z} &= z[-1 - z + \alpha x + \alpha y] \end{aligned}\right\} \quad (4.2.1.)$$

where α, β positive constants (α represents the biomass conversion rate in all cases), α is taken as the Hopf-bifurcation parameter.

The interior positive equilibrium $E^*(x^*, y^*, z^*)$ is given by

$$x^* = y^* = \frac{1+\beta}{1-\alpha+2\alpha\beta},\ z^* = \frac{3\alpha - 1}{1-\alpha+2\alpha\beta} \text{ with } \alpha > 1. \quad (4.2.2.)$$

Computing $p_1, p_2, p_3,\ p_1p_2 - p_3$, and then from $p_1p_2 - p_3 = 0$, we get

$$\alpha_c = \frac{\beta}{\beta - 2} \text{ with } \beta > 2. \quad (4.2.3.)$$

Solving the characteristic equation

$\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 = 0$, and $\alpha = \alpha_c$ we get the eigenvalues as

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$$\lambda_1 = -1,\ \lambda_{2,3} = \pm i w_0,\ \text{where } w_0 = \sqrt{\frac{\beta+1}{\beta-1}}. \qquad (4.2.4.)$$

$$\frac{d}{d\alpha}(p_1p_2 - p_3)_{\alpha=\alpha_c} = \frac{-(\beta-2)^2(\beta+1)\beta}{(\beta^2-1)^2} < 0, \text{ which implies,}$$

$$\frac{d}{d\alpha}(\lambda_1(\alpha))_{\alpha=\alpha_c} = \frac{1}{4}\frac{(\beta-2)^2}{(\beta^2-1)} > 0$$

See figure 4.2.1.

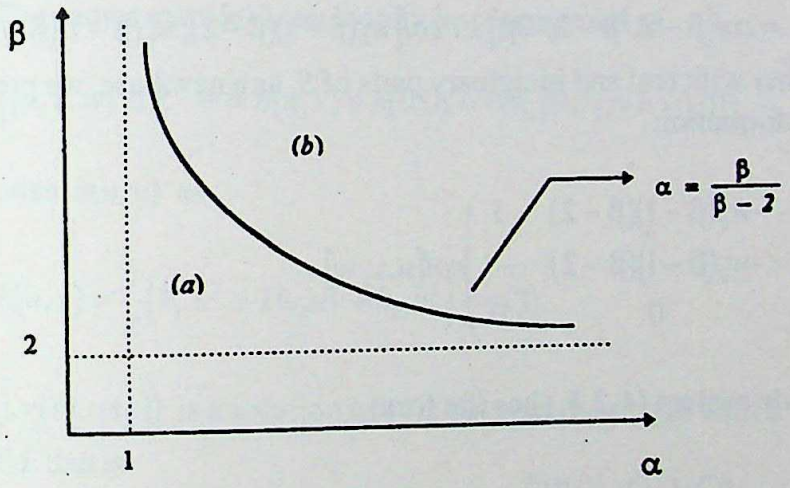

Figure 4.2.1.

The figure shows the stability status of (4.2.1.) with respect to (E^*) *as a function of the interaction parameters* α and β. *In the region (a), the positive equilibrium point (at which all the three species may coexist) is asymptotically stable. Hopf-bifurcating periodic solutions are obtained for all* α and β *lying on the curve in the region (b), the equilibrium is unstable.*

(i) The procedure as in [5] and [9]

Next we move the origin into the equilibrium $E^*(x^*, y^*, z^*)$ by co-ordinate transformation and performing the parameter transformation (4.2.3.) at the same time,

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we get the following system which is equivalent to the system (4.2.1.) :

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix}=\frac{1}{2(\beta-1)}\begin{pmatrix}-(\beta-2) & \beta & -\beta(\beta-2)\\ \beta & -(\beta-2) & -\beta(\beta-2)\\ \frac{2\beta}{\beta-2} & \frac{2\beta}{\beta-2} & -2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}+diag(x,y,z)\begin{pmatrix}-1 & \frac{\beta}{\beta-2} & -\beta\\ \frac{\beta}{\beta-2} & -1 & -\beta\\ \frac{\beta}{\beta-2} & \frac{\beta}{\beta-2} & -1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} \tag{4.2.5.}$$

The eigenvectors corresponding to the eigenvalues $\lambda_j(\alpha_c)\ j=1,2,3$ are

$S_1=col[1,-1,0],\ S_{2,3}=col[\beta-2,\ \beta-2,\ 2\beta]\pm i\ col[w_0(\beta-1)(\beta-2), w_0(\beta-1)(\beta-2),0].$
Introducing S_1 together with real and imaginary parts of S_3 as a new base, we perform the co-ordinate transformation

$$col[x,y,z]=\begin{pmatrix}\beta-2 & -w_0(\beta-1)(\beta-2) & 1\\ \beta-2 & -w_0(\beta-1)(\beta-2) & -1\\ 2\beta & 0 & 0\end{pmatrix}col[u,v,w].$$

In the new co-ordinate system (4.2.5.) has the form :

$$\begin{pmatrix}\dot{u}\\ \dot{v}\\ \dot{w}\end{pmatrix}=\begin{pmatrix}0 & -w_0 & 0\\ w_0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix}+\begin{pmatrix}F^1\\ F^2\\ F^3\end{pmatrix} \tag{4.2.6.}$$

where

$$\left.\begin{aligned}F^1&=-2w_0\beta uv\\ F^2&=w_0(\beta-2)u^2-\left(\frac{\beta^2(\beta-2)+4}{\beta-1}\right)uv-w_0(\beta-2)v^2+\frac{w^2}{w_0(\beta-1)(\beta-2)}\\ F^3&=-(\beta-2)(\beta+2)uw+w_0(\beta-2)^2vw\end{aligned}\right\} \tag{4.2.7.}$$

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The procedure given in [5] and in [9] is as follows;

The sign of the constant $C_1(\alpha_c)$ defined in terms of partial derivatives of (4.2.7.) determines the stability and direction of the bifurcation of the periodic solution. On calculation of $C_1(\alpha_c)$ (see Appendix), it is found that $\mathrm{Re}(C_1(\alpha_c)) = 0$, and hence no information concerning orbital stability of the bifurcating periodic solution.

(ii) Reduction of the System to the Center Manifold

We apply (as an alternative method) center manifold theorem which simply says that the orbital structure near $E^*(x^*, y^*, z^*)$ with α replaced by α_c is determined by the vector field (4.2.1.) restricted to the center manifold.

The center manifold can locally be represented as

$$\{(u,v,w) \in R^3 : w = h(u,v) \in h(0,0) = 0, h_u(0,0) = h_v(0,0)\} \qquad (4.2.8.)$$

we choose $h(u,v)$ as

$$h(u,v) = \frac{1}{2}\left[h_{11}u^2 + 2h_{12}uv + h_{22}v^2\right] + o(3).$$

If $(u(s), v(s), w(s))$ is a solution of (4.2.6.) near the origin with a value on the center manifold, that is

$w(s) = h(u(s), v(s))$, then

$$\dot{w}(s) - h_u(u,v)\dot{u} - h_v(u,v)\dot{v} \equiv 0 \qquad (4.2.9.)$$

Using (4.2.6.) in (4.2.9.), omitting terms of order at least three and equating coefficients to zero, we get

$h_{11} = h_{12} = h_{22} = 0$. This implies $w = h(u,v) = 0$.

Thus (4.2.6.) on center manifold becomes

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$$\left.\begin{aligned}\dot{u} &= -w_0 v - 2w_0\beta uv \\ \dot{v} &= w_0 u + w_0(\beta-2)u^2 - \left(\frac{\beta^2(\beta-2)+4}{\beta-1}\right)uv - w_0(\beta-2)v^2\end{aligned}\right\} \qquad (4.2.10.)$$

On calculating the constant "a" see [12] or the Index "I" (see Appendix), it is found that "a" (or I) = 0, and hence no information regarding orbital stability of the bifurcating periodic solution.

(iii) The Negrini-Salvadori Method.

As our third attempt to study orbital stability of the bifurcating periodic solution, we apply the method of Negrini-Salvadori [7] in which a Lyapunov function is assumed in the form :

$$V(u,v) = u^2 + v^2 + A(u,v) + B(u,v),$$

where A and B are homogeneous polynomials of order three and order four respectively.

$$A(u,v) = a_1u^3 + a_2u^2v + a_3uv^2 + a_4v^3$$

$$B(u,v) = b_1u^4 + b_2u^3v + b_3u^2v^2 + b_4uv^4 + b_5v^4.$$

The coefficient a_i and b_j are to be determined in such a way that the derivative of V with respect to the system (4.2.10.) is

$$\dot{V}(u,v) = G(u^2+v^2)^2 + o(5) \qquad (4.2.11.)$$

where the constant G is uniquely determined and the orbital stability of the bifurcating periodic solution depends on the sign of G.

It turns out (again) that $G = 0$, and hence (again) no information regarding orbital stability of the bifurcating periodic solution of (4.2.1.) can be deduced.

APPENDIX

(I) Calculation of $C_1(\alpha_c)$ based on Kazarinoff recipe-summary (see [5,9] Using (4.2.7.), i.e.

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$$F^1 = -2w_0\beta xy$$

$$F^2 = w_0(\beta-2)x^2 - \left[\frac{\beta^2(\beta-2)+4}{\beta-1}\right]xy - w_0(\beta-2)y^2 + \frac{z^2}{w_0(\beta-1)(\beta-2)}$$

$$F^3 = -(\beta-2)(\beta+2)xz + w_0(\beta-2)^2 yz$$

$$g_{11} = \left(F^1_{xx} + F^1_{yy}\right)/4 + i\left(F^2_{xx} + F^2_{yy}\right)/4 = 0$$

$$g_{02} = \left(F^1_{xx} - F^1_{yy} - 2F^2_{xy}\right)/4 + i\left(F^2_{xx} - F^2_{yy} + 2F^1_{xy}\right)/4 = \frac{\beta^2(\beta-2)+4}{2(\beta-1)} - 2iw_0$$

$$g_{20} = \left(F^1_{xx} - F^1_{yy} + 2F^2_{xy}\right)/4 + i\left(F^2_{xx} - F^2_{yy} + 2F^1_{xy}\right)/4 = \frac{\beta^2(\beta-2)+4}{2(\beta-1)} + 2iw_0(\beta-1)$$

$$g_{21} = \left[F^1_{xz} + F^2_{yz} + i\left(F^2_{xz} - F^1_{yz}\right)\right]\left[F^3_{xx} + F^3_{yy}\right]/4 + 2^{-1}\left[F^1_{xz} - F^2_{yz} + i\left(F^1_{yz} + F^2_{xz}\right)\right]$$

$$\left[F^3_{xx} - F^3_{yy} - 2iF^3_{xy}\right]/(4+2iw_0) = 0$$

$$C_1(\alpha_c) = i\left(g_{20}g_{11} - 2|g_{11}|^2 - 3^{-1}|g_{02}|^2\right)/2w_0 + g_{21}/2 =$$

$$i\left[-\frac{1}{3}\left(\frac{\beta^2(\beta-2)+4}{2(\beta-1)}\right)^2 + 4w_0^2\right]$$

Hence $\mathrm{Re}\, C_1(\alpha_c) = 0$

(II) Calculation of the constant "a"

(see [12, page 277])

$$a = \frac{1}{16}\left[f^1_{xxx} + f^1_{xyy} + f^2_{xxy} + f^2_{yyy}\right] + \frac{1}{16w_0}$$

$$\left[f^1_{xy}\left(f^1_{xx} + f^1_{yy}\right) - f^2_{xy}\left(f^2_{xx} + f^2_{yy}\right) - f^1_{xx}f^2_{xx} + f^1_{yy}f^2_{yy}\right]$$

where all partial derivatives are evaluated at the bifurcation point.

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Now $f^1 = -2w_0\beta xy$, $f^2 = w_0(\beta-2)x^2 - \left[\frac{\beta^2(\beta-2)+4}{\beta-1}\right]xy - w_0(\beta-2)y^2$.

Thus $a = 0$.

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PERSISTENCE IN A 3-SPECIES PREDATOR-PREY MODEL

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ABSTRACT

The ecological notion of system persistence in modeling the interaction of two competing predatory populations living exclusively on a common prey, is investigated.

Freedom and Waltman [4], and El-Owaidy and Ammar [2] have discussed the persistence of such models based upon the assumption of nonexistence of limit cycles. In this paper, however; the nonexistence of limit cycles is proven, first, by global asymptotic stability of equilibria through the construction of a suitable Lyapunov function, and second, persistence criteria of the system are obtained.

Key words: Predator-prey system, Lyapunov function, limit cycle, asymptotic global stability.

Mathematics Subject Classification (2000) : 92 B05

INTRODUCTION

One of the most interesting questions in ecological models concerns the survival of all species of the model. This question of persistence becomes more important when two or more species compete for a single prey species.

The significant concepts of permanence and persistence, both exclude extinction of species for all positive initial conditions. In biological terms, persistence means that the density of each population remains, asymptotically, above a positive bound independent of the initial conditions, i.e.all species stay away from extinction.

Mathematically, this may be stated in terms of behaviour of solutions of the model which represents the biological phenomenon.

An ecological differential system

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$\dot{x}_i = x_i f(x_1, x_2, \ldots x_n)$ for $i = 1,2,\ldots,n$

is said to be permanent or uniformly persistent if there exists a compact set k in the interior of $R_+^n = \{x \in R^n : x_i \geq 0$ for $1 \leq i \leq n$ such that all orbits end up in k. This guarantees that the number of each population $x_i(t)$ is bounded away from zero if $x_i(0) >$ for all i.

By the term weak persistence, we mean that for all i, $\limsup_{t \to +\infty} x_i(t) > 0$ whenever $x(0)>0$. Under this "*lim sup*" definition of persistence, a population can frequently approach extinction. By the term strong persistence, we mean that for all i, $\liminf_{t \to +\infty} x_i(t) > 0$ whenever $x(0)>0$.

We have applied the latter definition of persistence as used in Freedman and Waltman [4]. This difinition of persistence was reformulated in [4] as follows :

"A persistence with initial conditions in the positive cone will persist if there are no $\omega -$ *limit* points of the solution on the boundary of the positive cone".

This means if $\square(X)$ be the orbit through the point $X = (x, y, z)$ with $x>0, y>0, z>0$, and if $\Omega(X)$ be the $\omega -$ *limit* set of $\square(X)$, then $\Omega(X)$ be interior to the positive cone.

Since, the question of two predator populations competing for a single prey population has occupied an important place in ecological literature, so in this paper, we have considered a system modeling the interactions of two competing predator populations living exclusively on a common prey. For both the predator populations we have taken density-dependent death rates and in this sense, we have extended the result of El-Owaidy and Ammar [2].

Freedman and Waltman [4] in theorem 4.1., obtained persistence criteria by an assumption that for such a system, there are no limit cycles surrounding an interior equilibrium in a co-ordinate plane.

We have actually proved the non-existence of limit cycles by constructing Lyapunov functions for the equilibria and have obtained conditions under which the equilibria are are globally asymptotically stable. This implies that all the trajectories of the system in the positive cone will spiral toward this equilibrium point, that is non-

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existence of limit cycles.

Then, we have applied the persistence criteria as given in theorem 4.1. under reference, to obtain conditions for persistence of our system. Thus our result of persistence seems to be more comprehensive than that in theorem 4.1. of Freedman and Waltman [4].

The organization of this paper is as follows :

In section 2, we give the model. Section 3 deals with the global asymptotic stability of the problem. Section 4 deals with persistence results and we illustrate these by an example.

THE MODEL AND EQUILIBRIA

A system modeling the interaction of two competing predator populations $y(t)$ and $z(t)$ living exclusively on a common prey $x(t)$.

$$\dot{x} = xf(x) - yp(x) - zq(x)$$

$$\dot{y} = y[-g(y) + c_1 p(x)] \tag{2.1}$$

$$\dot{z} = z[-h(z) + d_1 q(x)],$$

$$x(0) = x_0 > 0, y(0) y_0 \geq 0, z(0) = z_0 \geq 0.$$

Where $\left(\cdot = \frac{d}{dt}\right)$, c_1, d_1 are positive constants, known as food conversion rates.

We make the following assumptions, which are consistent with models of predator-prey systems.

(H1) : $f(x)$; specific growth rate of prey x

$f(0)>0, f'(x) \leq 0$ for all $x > 0$, there exists $k > 0$ (the carrying capacity) such that $f(x) > 0$ on $0 \leq x < k, f(k) = 0$ and $f(x) < 0$ on $x > k$.

2) : $p(x)$; the functional response of the predator y with respect to the prey x and $p(0) = 0, p'(x) > 0$ for all $x \geq 0$.

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3) : $q(x)$; the functional response of the predator z with respect to the prey x and $q(0) = 0$, $q'(x) > 0$ for all $x \geq 0$.

(H4) : we assume $q(x) = \alpha p(x)$, where α is a positive constant.

5) : $g(y)$; density-dependent death rate of the predator y and

$g(y)$; $g'(y) \geq 0$ for all $y > 0$.

(H6) : $h(z)$; density-dependent death rate of the predator z and

$h(0) > 0$, $h'(z) \geq 0$ for all $z > 0$.

Clearly the system (2.1) has equilibiruim point $E_0(0, 0, 0)$. By assumption (H1), $E_1(k, 0, 0)$ is also an equilibrium point. We assume that each of the predators y and z can survive on the prey x, that is there exist equilibirium points $\overset{*}{E}\left(\overset{*}{x}, \overset{*}{y}, 0\right)$ and $\hat{E}(\hat{x}, 0, \hat{z})$ such that

$$\overset{*}{x} f\left(\overset{*}{x}\right) - \overset{*}{y}\, p\left(\overset{*}{x}\right) = 0 \tag{2.2}$$

$$-g\left(\overset{*}{y}\right) + c_1 p\left(\overset{*}{x}\right) = 0$$

and

$$\hat{x} f(\hat{x}) - \hat{z} q(\hat{x}) = 0$$

$$-h(\hat{z}) + d_1 q(\hat{x}) = 0 \tag{2.3}$$

where $\overset{*}{x}, \overset{*}{y}, \hat{x}, \hat{z} > 0$ and $\overset{*}{x} < k, \hat{x} < k$. (2.4)

GLOBAL ASYMPTOTIC STABILITY OF EQUILIBRIA $\overset{*}{E}\left(\overset{*}{x}, \overset{*}{y}, 0\right)$ and $\hat{E}(\hat{x}, 0, \hat{z})$.

3.1 Assume (H1)-(H6) hold for the system (2.1) and in a neighbourhood of $\left(\overset{*}{x}, \overset{*}{y}, 0\right)$ in the positive cone, the function $\dfrac{xf(x)}{p(x)}$ is strictly decreasing.

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Then the equilibrium point $\overset{*}{E}\left(\overset{*}{x},\overset{*}{y},0\right)$ is globally asymptotically stable.

Proof :

Define a Lyapunov function $V(x,y,z)$ as [see [1]] :

$$V(x,y,z)=\int_{\overset{*}{x}}^{x}\left[c_1\left(1-\frac{p(\overset{*}{x})}{p(w)}\right)+d_1\left(1-\frac{q(\overset{*}{x})}{q(w)}\right)\right]dw+\int_{\overset{*}{y}}^{y}\left(\frac{w-\overset{*}{y}}{w}\right)dw+z \qquad (3.1)$$

Now $\left(\overset{*}{x},\overset{*}{y},0\right)=0$ and due to (H2) and (H3), (x, y, z) is positive in the region:

$0<x<\overset{*}{x}<k,\ 0<\overset{*}{y}<y<\beta_1,\ 0<z<\beta_2,\ \beta_1,\ \beta_2$ are positive constants.

$$\frac{d}{dt}V(x,y,z)=c_1\left[\dot{p}(x)-p\left(\overset{*}{x}\right)\right]\left[\frac{xf(x)}{p(x)}-\frac{zq(x)}{p(x)}-y\right]$$

$$+d_1\left[q(x)-q\left(\overset{*}{x}\right)\right]\left[\frac{xf(x)}{q(x)}-\frac{yp(x)}{q(x)}-z\right]$$

$$+\left(y-\overset{*}{y}\right)\left[-g(y)+c_1p(x)\right]+z\left[-h(z)+d_1q(x)\right]$$

using (2.2), (H4) and with some algebraic manipulations we get

$$\dot{V}(x,y,z)=\left[c_1\left(p(x)-p\left(\overset{*}{x}\right)\right)+\frac{1}{\alpha}d_1\left(q(x)-q\left(\overset{*}{x}\right)\right)\right]\left[\frac{xf(x)}{p(x)}-\frac{\overset{*}{x}f\left(\overset{*}{x}\right)}{p\left(\overset{*}{x}\right)}\right]$$

$$+\left(y-\overset{*}{y}\right)\left[\left(g\left(\overset{*}{y}\right)-g(y)\right)+\frac{d_1}{\alpha}\left(q\left(\overset{*}{x}\right)-q(x)\right)\right]$$

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$$+z\left[\alpha c_1\left(p\left(\overset{*}{x}\right)-p(x)\right)+\left(h(0)-h(z)\right)\right]<0$$

Thus $\overset{*}{E}\left(\overset{*}{x},\overset{*}{y},0\right)$ is globally asymptotically stable.

3.2. Assume (H1)-(H6) hold for the system (2.1) and in a neighbourhood of $(\hat{x},0,\hat{z})$ in the positive cone, the function $\dfrac{xf(x)}{p(x)}$ is strictly decreasing. Then the equilibrium point $\hat{E}(\hat{x},0,\hat{z})$ is globally asymptotically stable.

Proof :

Define Lypunov function (x, y, z) as :

$$V(x,y,z)=\int_{\hat{x}}^{x}\left[c_1\left(1-\frac{p(\hat{x})}{p(w)}\right)+d_1\left(1-\frac{q(\hat{x})}{q(w)}\right)\right]dw+y+\int_{\hat{z}}^{z}\left(\frac{w-\hat{z}}{w}\right)dw \qquad (3.2)$$

Rest of the proof follows as in Lemma 3.1.

Remark 3.1.

We consider equilibirium points $E_0(0,0,0)$ and $E_1(k,0,0)$.

The eigenvalues of the variational matrix (E_0) of the system (2.1.) about $E_0(0,0,0)$ are:

$$\lambda_1=f(0)>0,\ \lambda_2=-g(0)<0,\ \text{and}\ \lambda_3=-h(0)<0. \qquad (3.3)$$

Clearly $E_0(0,0,0)$ is a hyperbolic point and is unstable along the *x-axis*. This implies that the prey population x grows near E_0.

The eigenvalues of the variational matrix (E_1) of the system (2.1) about $E_1(k,0,0)$ are :

$$\lambda_1=kf'(k)<0,\ \lambda_2=-g(0)+c_1p(k),\ \lambda_3=-h(0)+d_1q(k) \qquad (3.4)$$

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Thus $E_1(k,0,0)$ is asymptotically stable along the *x-axis*. This implies that the prey population x remains in neighbourhood of k.

Remark 3.2.

For existence of $\overset{*}{E}\left(\overset{*}{x},\overset{*}{y},0\right)$ and $\hat{E}(\hat{x},0,\hat{z})$ it is necessary that

$$g(0)+c_1p(k)>0 \text{ and } -h(0)+d_1q(k)>0 \tag{3.5}$$

as it implies increase of predator population y and predator population z.

PERSISTENCE CRITERIA

In section 3, we have given necessary conditions for existence of equilibria $\overset{*}{E}\left(\overset{*}{x},\overset{*}{y},0\right)$ and $\hat{E}(\hat{x},0,\hat{z})$, and creiteria for their global asymptotic stability.

In this section, we shall assume global stability of these equilibria and obtain persistence criteria for the system (2.1.).

first, we prove the following two lemmas.

Lemma 4.1. The equilibrium $\overset{*}{E}\left(\overset{*}{x},\overset{*}{y},0\right)$ in the interior of the *x-y* plane is unstable in the positive direction orthogonal to *x-y* plane if

$$-h(0)+d_1q\left(\overset{*}{x}\right)>0 \text{ or } -h(0)+d_1\alpha p\left(\overset{*}{x}\right)>0.$$

Proof :

The proof is immediate upon computing the variational matrix $\left(\overset{*}{E}\right)$ of system (2.1) about $\overset{*}{E}\left(\overset{*}{x},\overset{*}{y},0\right)$.

We have :

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$$V\left(\overset{*}{E}\right)=\begin{pmatrix} f\left(\overset{*}{x}\right)+\overset{*}{x}f'\left(\overset{*}{x}\right)-\overset{*}{y}p'\left(\overset{*}{x}\right) & -p\left(\overset{*}{x}\right) & -q\left(\overset{*}{x}\right) \\ -gc_1\overset{*}{y}p'\left(\overset{*}{x}\right) & \left(\overset{*}{y}\right)+c_1p\left(\overset{*}{x}\right)-\overset{*}{y}g'\left(\overset{*}{y}\right) & 0 \\ 0 & 0 & -h(0)+d_1q\left(\overset{*}{x}\right) \end{pmatrix}.$$

This if $-h(0)+d_1q\left(\overset{*}{x}\right)>0$ or $-h(0)+d_1\alpha p\left(\overset{*}{x}\right)>0$.

we have the result.

Lemma 4.2. The equilibrium $\hat{E}(\hat{x},0,\hat{z})$ in the interior of the *x-z* plane is unstable in the positive direction orthogonal to *x-z* plane if $-g(0)+c_1p(\hat{x})>0$.

Proof : The proof is immediate upon computing the variational matrix $V\left(\hat{E}\right)$ of the system (2.1) about $\hat{E}(\hat{x},0,\hat{z})$.

Now, to apply persistence criteria to our system (2.1), we have to check hypotheses (B1)-(B-4) of theorem 4.1. in Freedman and Waltman [4] and boundedness of solutions.
we have :

$$F(x,y,z)=f(x)-y\frac{p(x)}{x}-z\frac{q(x)}{x}$$

$$G_1(x,y,z)=-g(y)+c_1p(x)$$

$$G_2(x,y,z)=-h(z)+d_1q(x),$$

therefore; condition (B1) is trivially satisfied due to (H1)-(H6). Also notice that

$$p(0)=0, \text{ and } p'(x)>0$$

implies $p(x)$ is a strictly increasing positive function. Similarly $q(x)$.
Condition (B2) is true due to (H1).

$$F(0,0,0)=f(0)>0,\ F(k,0,0)=f(k)=0$$

$$\frac{\partial f}{\partial x}(x,0,0)=f'(x)\le 0.$$

Satisfying (B3) there are no equilibria on *y-axis* or *z-axis* in *y-z* plane, for if we

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suppose that there exists an equilibrium $E(0, y_1, z_1)$ in y-z plane which is given by : $g(y_1) = 0$ and $h(z_1) = 0$, then this contradicts (H5) and (H6), and satisfying (B4) each predator can survive on prey, that is, there exist points $\overset{*}{E}\left(\overset{*}{x}, \overset{*}{y}, 0\right)$ and $\hat{E}(\hat{x}, 0, \hat{z})$ such that (2.2) and (2.3) hold.

Also see remark 3.2. Moreover, we require;

5) : Boundedness of solutions of system (2.1.). We suppose that the functions $f(x)$, $p(x)$, $q(x)$, $g(y)$ and $h(z)$ are sufficiently smooth so that the solution to the initial value problem (2.1) exists, is unique and continuable for positive values of t. Regarding boundedness of solution, see Freedman and Waltman [3] and [4].

We state and prove the main theorem.

Theorem 4.1 Let (B1)-(B5) hold. Let $\dfrac{xf(x)}{p(x)}$ be a strictly decreasing function and

$$-g(0) + c_1 p(\hat{x}) > 0 \text{ and } -h(0) + d_1 \alpha p\left(\overset{*}{x}\right) > 0 \tag{4.1}$$

Then the system (2.1.) persists.

Proof : By (B5) solutions are bounded.

By remark 3.1 the equilibrium $E_0(0,0,0)$ is unstable along the *x-axis* and unstable manifold of $E_1(k,0,0)$ is two dimensional. Conditions (4.1) follow from lemmas 4.1. and 4.2. Non-existence of limit cycles follows from lemmas 3.1. and 3.2.

This completes the proof.

Remark 4.1.

From (H4), we have

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$$\alpha = \frac{q(x)}{p(x)} = \frac{\text{rate of prey consumption per predator } z \text{ at prey density } x}{\text{rate of prey consumption per predator } y \text{ at prey density } x}$$

Thus, if we consider α as a parameter then, the system (2.1.) will persit provided

$$\alpha > \frac{h(0)}{d_1 p\left(\overset{*}{x}\right)}$$

Remark 4.2.

We have discussed persistence criteria for a system modeling the interaction of two competing predator populations living exclusively on a common prey. But in the same way, the persistence criteria can be obtained for the system modeling interactions between two prey populations and one predator population, that is,

$$\begin{aligned} \dot{x} &= xf(x) - zq(x) \\ \dot{y} &= yg(y) - zr(y) \\ \dot{z} &= z\left[-h(z) + d_1 q(x) + d_2 r(y)\right] \end{aligned} \tag{4.2.}$$

and for three-level food web, that is

$$\begin{aligned} \dot{x} &= xf(x) - yp(x) - zq(x) \\ \dot{y} &= y\left[-g(y) + c_1 p(x)\right] - zr(y) \\ \dot{z} &= z\left[-h(z) + d_1 q(x) + d_2 r(y)\right] \end{aligned} \tag{4.3}$$

For construction of Lyapunov functions for the systems (4.2) and (4.3) see [1].

Example 4.1.

To illustrate the theorem 4.1., consider the system with the Holling type II functional response.

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$$\dot{x} = x(1-x) - y\frac{2x}{1+x} - z\alpha\frac{2x}{1+x}$$

$$\dot{y} = y\left[-(1+y) + \frac{33}{16}\frac{2x}{1+x}\right] \tag{4.4}$$

$$\dot{z} = z\left[-(1+z) + \frac{11}{9}\alpha\frac{2x}{1+x}\right]$$

Here, $k = 1, \dfrac{xf(x)}{p(x)} = \dfrac{1-x^2}{2}$

Thus $\dfrac{xf(x)}{p(x)}$ is a strictly decreasing function for x>0.

$$\overset{*}{E}\left(\overset{*}{x}, \overset{*}{y}, 0\right) = \overset{*}{E}\left(\frac{1}{2}, \frac{3}{8}, 0\right)$$

$$p\left(\overset{*}{x}\right) = p\left(\frac{1}{2}\right) = \frac{2}{3}$$

$\alpha > \dfrac{h(0)}{d_1 p\left(\overset{*}{x}\right)} = \dfrac{27}{22}$. Thus take $\alpha = 2$

$$\hat{E}(\hat{x}, 0, \hat{z}) = \hat{E}\left(\frac{1}{3}, 0, \frac{2}{9}\right)$$

To check condition (4.1)

$$-g(0) + c_1 p(\hat{x}) = -1 + \frac{33}{16} \times \frac{1}{2} = \frac{1}{32} > 0$$

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$$-h(0)+d_1\alpha p\left(\overset{*}{x}\right)=-1+\frac{11}{9}\times 2\times\frac{2}{3}=-1+\frac{44}{27}=\frac{17}{27}>0$$

Theorem 4.1. applies and hence the system (4.4.) is persistent.

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ON KÖTHE-TOEPLITZ DUAL OF A CERTAIN SEQUENCE SPACE AND MATRIX-TRANSFORMATIONS

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(Revised 31.12.2002)

ABSTRACT

We extend the well known sequence space $l(p)$ to $l(p,\upsilon)$ for $\upsilon \geq 0$ and find its Köthe-Toeplitz dual and continuous dual. Some matrix transformations have also been considered.

INTRODUCTION

Let l_∞ and c be the Banach spaces of bounded and convergent sequences $x=(x_k)$ with usual norm $\|x\| = \sup_k |x_k|$. For $p=(p_k)$ with $p_k > 0$ and $\sup p_k < \infty$ Maddox [1] defined the space

$$l(p) = \left\{ x=(x_k) : \sum_k |x_k|^{p_k} < \infty \right\};$$

and it was shown that the continuous dual of $l(p)$, i.e. $l^*(p)$ could be identified with $l(q)$, where $(1/p_k)+(1/q_k)=1$

Further, in [2] the Köthe-Toeplitz dual of $l(p)$ was determined.

In this paper, we extend the space $l(p)$ to $l(p,\upsilon)$ for $\upsilon \geq 0$ and determine the continuous and Köthe-Toeplitz duals of $l(p,\upsilon)$. We also characterize the matrices of classes $(l(p,\upsilon), l_\infty)$ and $(l(p,\upsilon), c)$.

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Definition :

1.1 For $p=(p_k)$ with $p_k>0$ and $\sup p_k<0$, we define for $\upsilon \geq 0$.

$$l(p,\upsilon)=\left\{x=(x_k):\sum_k k^{-\upsilon}|x_k|^{p_k}<\infty\right\}$$

where $l(p)=\left\{x=(x_k):\sum_k |x_k|^{p_k}<\infty\right\}$

and $M(q,\upsilon)=\left\{a=(a_k):\sum_k k^{\upsilon(q_k^{-1})}|a_k|^{q_k}E^{-q_k/p_k}<\infty\right\}$

where $E>1$ and $1/p_k+1/q_k=1$

1.2 Paranormed Space :

A paranorm g defined on a linear space X, is a function $g:X\to R$ having the following properties.

(i) $g(\theta)=0$ where θ is the zero element in X.

(ii) $g(x)=g(-x)$ for all $x \in X$.

(iii) $g(x+y)\leq g(x)+g(y)$ for all $x,y \in X$.

(iv) The scalar multiplication is continuous, that is $\lambda_n \to \lambda(n\to\infty)$ &

$g(x_n-x)\to 0$ as $n\to\infty$ for, $\lambda_n, \lambda \in C$ and $x_n, x \in X$ imply that

$g(\lambda_n x_n-\lambda x)\to 0(n\to\infty)$.

A paranormed space is a linear space X together with a paranorm g. A total paranorm is a paranorm such that $g(x)=0$ implies $x=0$.

It is easy to see that $l(p,\upsilon)$ is paranormed space by

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$$h(x)=\left[\sum k^{-\upsilon}|x_k|^{p_k}\right]^{\frac{1}{M}}$$

where $M=\max(1,\ \sup\ p_k)$.

For; (i) $h(x)=\left[\sum k^{-\upsilon}|x_k|^{p_k}\right]^{\frac{1}{M}}$

$=0$ if and only if $x=0$

(ii) $h(-x)=\left[\sum k^{-\upsilon}|-x_k|^{p_k}\right]^{\frac{1}{M}}$

$=\left[\sum k^{-\upsilon}|x_k|^{p_k}\right]^{\frac{1}{M}}$

$=h(x)$

(iii) $h(x+y)=\left[\sum k^{-\upsilon}|x_k+y_k|^{p_k}\right]^{\frac{1}{M}}$

By minkowski's inequality we have

$$h(x+y)\le\left[\sum k^{-\upsilon}|x_k|^{p_k}\right]^{\frac{1}{M}}+\left[\sum k^{-\upsilon}|y_k|^{p_k}\right]^{\frac{1}{M}}$$

$=h(x)+h(y)$

Hence $h(x+y)\le h(x)+h(y)$

(iv) If $\lambda_n,\ \lambda_0\ \in C$ with $\lambda_n\to\lambda_0\ (n\to\infty)$ and if $x_n,\ a\ \in l(p,\upsilon)$ with

$x_n\to a\ (n\to\infty)$ in the sense that $h(x_n-a)\to 0(n\to\infty)$

& $h(\lambda_n x_n-\lambda_n a)=\left[\sum_n n^{-\upsilon}|\lambda_n x_n-\lambda_0 a|^{p_n}\right]^{\frac{1}{M}}$

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$$= \left[\sum_n n^{-\upsilon} \left| (\lambda_n - \lambda_0)(x_n - a) + \lambda_0 (x_n - a) + (\lambda_n - \lambda_0) a \right|^{p_n} \right]^{\frac{1}{M}}$$

$$\leq \left[\sum n^{-\upsilon} \left| (\lambda_n - \lambda_0)(x_n - a) \right|^{p_n} \right]^{\frac{1}{M}} + \left[\sum n^{-\upsilon} \left| \lambda_0 (x_n - a) \right|^{p_n} \right]^{\frac{1}{M}}$$

$$+ \left[\sum n^{-\upsilon} \left| (\lambda_n - \lambda_0) a \right|^{p_n} \right]^{\frac{1}{M}}$$

Now $\left|(\lambda_0)\right|^{p_n} \leq \max(1, |\lambda_0|) = H(\text{say})$

Hence, $h(\lambda_n x_n - \lambda_0 a) \to 0 (n \to \infty)$

So, $l(p, \upsilon)$ is a total paranormed space.

In the next section we prove that $l^*(p, \upsilon) = M(q, \upsilon)$ i.e. the Köthe-Toeplitz dual of $l(p, \upsilon)$ is $M(q, \upsilon)$; and $l^*(p, \upsilon)$ is isormorphic to $M(q, \upsilon)$.

CONTINUOUS AND KÖTHE-TOEPLITZ DUALS

If (X, h) is a paranormed space with paranorm h then we denote X^* the continuous dual of X that is the set of all continuous linear functionals on X. If E is a set of complex sequences $x = (x_k)$ then E^+ will denote the generalized Köthe-Toeplitz dual of E.

$$E^+ = \left\{ a = (a_k) : \sum_k a_k x_k \text{ converges for all } x \in E \right\}$$

Theorem :

2.1 Let $1 < p_k \leq \sup p_k < \infty$ for all k. Then for $\upsilon \geq 0$ $l^*(p, \upsilon) = M(q, \upsilon)$.

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Proof :

Here we use the same analysis as in [2]. Let $x \in l(p,\upsilon)$ and $a \in M(q,\upsilon)$. Therefore

$$\sum_k |a_k x_k| \le \sum_k |a_k| k^{\upsilon/p_k} . |x_k| k^{-\upsilon/p_k}$$

$$\le \sum_k |a_k|^{q_k} k^{\upsilon q_k/p_k} . E^{-q_k/p_k} + E \sum_k |x_k|^{p_k} k^{-\upsilon}$$

$$\le \sum_k |a_k|^{q_k} k^{\upsilon(q_k-1)} E^{-q_k/p_k} + E \sum_k |x_k|^{p_k} k^{-\upsilon}$$

$$< \infty$$

Hence $\sum_k a_k x_k$ is convergent. Therefore $a \in l^*(p,\upsilon)$.

Conversely, let $\sum_k a_k x_k$ converges and $x \in l(p,\upsilon)$. This implies that $a \in M(q,\upsilon)$, for otherwise we can determine integers.

$$0 = n(0) < n(1) < n(2) < \ldots\ldots$$

such that

$$M_j = \sum_{k=n(j-1)+1}^{n(j)} k^{\upsilon\left(q_k^{-1}\right)} |a_k|^{q_k} (j+1)^{-q_k/p_k} > 1$$

Define a sequence $x = (x_k)$ by

$$x_k = (sign\ a_k) |a_k|^{q_k^{-1}} (j+1)^{-q_k} M_j^{-1} k^{\upsilon\left(q_k^{-1}\right)}, \text{ for } n(j-1) \le k \le n(j)$$

$$= 0, \text{ otherwise.}$$

Therefore

$$\sum_{k=n(j-1)+1}^{n(j)} k^{-\upsilon} |x_k|^{p_k} = \sum_{k=n(j-1)+1}^{n(j)} k^{-\upsilon} k^{\upsilon\left(q_k^{-1}\right)^{p_k}} |a_k|^{\left(q_k^{-1}\right)p_k} (j+1)^{-q_k/p_k} M_j^{-p_k}$$

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$$\leq (j+1)^{-2} M_j^{-1} \sum_{k=n(j-1)+1}^{n(j)} k^{-q_k^{-1}\upsilon_k \upsilon\left(q_k^{-1}\right)^{p_k}} |a_k|^{q_k} (j+1)^{-q_k/p_k}$$

$$\leq (j+1)^{-2}$$

i.e. $x \in l(p,\upsilon)$. But

$$\sum_{k=n(j-1)+1}^{n(j)} a_k x_k = (j+1)^{-1}$$

So that $\sum_k a_k x_k$ diverges. Hence a contradictioin and therefore $a \in M(q,\upsilon)$.

Theorem :

2.2 Let $1 < p_k \leq \sup p_k < \infty$. Then $l^*(p,\upsilon)$ is isomorphic to $M(q,\upsilon)$.

Proof :

It is easy to see that $x \in (p,\upsilon)$ can be written as $x = \sum_k x_k e_k$ where $e_k = (0,0,...,$ $0,1,0,...)$. Therefore for any $f \in l^*(p,\upsilon)$.

$f(x) = \sum_k x_k f(e_k) = \sum_k x_k a_k$, where $f(e_k) = a_k$.

By theorem 2.1, $a \in M(q,\upsilon)$ if $\sum_k a_k x_k$ converges for $x \in l(p,\upsilon)$.

Now, if $x \in l(p,\upsilon)$ and $a \in M(p,\upsilon)$; $\sum_k a_k x_k$ converges and defines a linear functional on $l(p,\upsilon)$. Furthermore, it is easy to check that

$$\sum_{k=1}^{\infty} |a_k x_k| \leq E\left(\sum_k k^{\upsilon\left(q_k^{-1}\right)} |a_k| E^{\frac{-q_k}{p_k}} + 1\right) g(x)$$

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whenever $g(x) \le 1$. Therefore $\sum_k a_k x_k$ defines an element of $l^*(p,\upsilon)$. Obviously, the map. $T : l^*(p,\upsilon) \to M(q,\upsilon)$ given by $T(f) = (a_1, a_2 .....)$ is linear and bijective. Hence $l^*(p,\upsilon)$ is isomorphic to $M(q,\upsilon)$.

MATRIX TRANSFORMATIONS

Let $A = (a_{nk})$ be an infinite matrix of complex numbers $a_{nk}(n,k = 1,2,.....)$ and X, Y be two subsets of the space of complex sequences. We say that the matrix A defines a matrix transformation from X into Y, if for every sequence $x = (x_k) \in X$ the sequence $Ax = (y_n) \in Y$, where $Y_n = \sum a_{nk} x_k$ provided the series is convergent. The class of all such matrix transformations from X into Y will be denoted by (X, Y).

In this section we characterize the matrices of the classes $(l(p,\upsilon), l_\infty)$ and $(l(p,\upsilon), c)$.

Theorem :

3.1 Let $1 < p_k \le \sup p_k < \infty$. Then $A \in (l(p,\upsilon), l_\infty)$ if and only if there exists an integer $E>1$ such that $U_n(E) < \infty$ for every n, where

$$U_n(E) = \sum_k k^{\upsilon(q_k^{-1})} |a_{nk}|^{q_k} E^{-q_k/p_k} < \infty .$$

Proof :

Necessity : Let $A \in (l(p,\upsilon), l_\infty)$ and $x \in l(p,\upsilon)$. Therefore $\sum_k a_{nk} x_k < \infty$ for each n and for $x \in l(p,\upsilon)$. Hence $(a_{nk})_k \in l^*(p,\upsilon)$ for each n. Since $l^*(p,\upsilon)$ is isomorphic to $l^*(p,\upsilon)$. Therefore $A_n(x) = \sum a_{nk} x_k$ is an element of $l^*(p,\upsilon)$. We

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know that $l(p,\upsilon)$ and l_∞ both are complete and by wilansky [3] that there exists a constants K such that

$$\|A_n(x)\| \leq K\|x\|$$

Hence by the principle of condensation of singularities [4]

$U_n(E) < \infty$, for every n.

Sufficiency : Let $x \in l(p,\upsilon)$ and the condition $U_n(E) < \infty$ hold for every n. Therefore, for $E > 1$.

$$\sum_k |a_{nk}x_k| \leq \sum_k k^{\upsilon(q_k^{-1})}|a_{nk}|^{q_k} E^{-q_k/p_k} + E\sum_k |x_k|^{pk} k^{-\upsilon} < \infty$$

Hence $A \in (l(p,\upsilon), l_\infty)$.

This completes the proof of the theorem.

Theorem :

3.2 Let $1 < p_k \leq \sup p_k < \infty$. Then $A \in (l(p,\upsilon), c)$ if and only if

(i) for $E > 1$. $U_n(E) < \infty$ for every n; and

(ii) $a_{nk} \to \alpha_k (n \to \infty)$ for every k.

Proof :

Necessity : Condition (i) follows as in Theorem 3.1; and condition (ii) follows from the fact that $e_k \in l(p,\upsilon)$;

Sufficiency : Let the conditions (i) and (ii) hold and that $x \in l(p,\upsilon)$. Therefore as in Thbeorem 3.1, we have

$$\sum_k |a_{nk}x_k| < \infty,$$

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i.e. $\sum_k a_{nk} x_k$ is absolutely convergent for

each n and for $x \in l(b,\upsilon)$. Moreover, we can choose an integer $k_0 > 1$ such that

(1) $$g_{k_0}(x) = \sum_{k=k_0}^{\infty} k^{-\upsilon} |x_k|^{p_k} < 1$$

Therefore by theorem 2.2 we have

(1) $$\sum_{k \geq k_0} |a_{nk} - d_k \| x_k| \leq E \left(\sum_{k \geq k_0} k^{\upsilon(q_k^{-1})} |a_{nk} - \alpha_k|^{q_k} E^{-q_k/p_k} + 1 \right) \left(g_{k_0}(x) \right)^{1/M}$$

where $M = \max(1, \sup p_k)$. Clearly (1) holds if $g_{k_0}(x) = 0$.

Since

$$\sum_{k \geq k_0} k^{\upsilon(q_k^{-1})} |a_{nk} - \alpha_k|^{q_k} E^{-q_k/p_k} \leq 2U_n(E) < \infty$$

Therefore by (1) we have

$\sum_k a_{nk} x_k \to \sum_k \alpha_k x_k (n \to \infty)$, and hence $A \in (l(p,\upsilon), c)$.

This completes the proof of the theorem.

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INITIAL VALUE ABELIAN THEOREM FOR DISTRIBUTIONAL MODIFIED STIELTJES TRANSFORMATIONS

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ABSTRACT

In the first part we define the space M′(r) and modified Stielfjes transformation introduced by Lavoiene & Misra [1] and Marichev [8] respectively. In the second part we use quasi-asymptotic behaviour of distribution to prove initial value Abelian theorem.

Key Words and Phrases : Modified Stieltjes transformation of distribution, quasi-asymptotic behaviour of of distributions.

AMS Subject Classification Codes (1991) : 44 and 46 F.

INTRODUCTION

The notion of quasi-asymptotic behaviour of tempered distributions was introduced by Zavialov [4]. Here we give, first several assertions concerning the quasi-asymptotic behaviour at 0^+ for elements from S'_+. The motivation for our investigation made in the first part are theorems of Abelian type for the distributional modified Stieltjes transformation, while we shall see in the second part that the quasi-asymptotic behaviour at 0^+ is more convenient than the ordinary asymptotic behaviour of a disbritution especially to prove Abelian type theorems for the modified Stieltjes transformation.

NOTATIONS AND DEFITIONS

N is the set of natural numbers and $N_0 = N \cup \{0\}$. R is the Set of real

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numbers and $\mathbb{C}$ is the set of complex numbers. S'_+ is the space of temperecd distributions with support in the $(0,\infty)$. S' is the space of distribution of slow growth or tempered distributions which is the dual of S. We set $R_+ = \{x \in R : x > 0\} = (0,\infty)$ and its closure in R is denoted by R_+^-, $R_-^- = (-\infty, 0]$. We denote the partial differential operator by D^p, $p \in N_0$.

Definition 1[3] :

A function $\rho : (a,\infty) \to R$, $a \in R$ is called regularly varying function at infinity if it is positive, measurable and there exists a real number α such that for each $x > 0$

$$\lim_{k\to\infty} \frac{\rho(kx)}{\rho(k)} = x^{\alpha}.$$

the number α is called the index of ρ.

Definition 2 :

If function L is regularly varying function with index $\alpha = 0$ then it is called slowly varying function at infinity. Thus L is the positive, measurable on (a,∞) from definition 1, we have

$$\lim_{k\to\infty} \frac{L(kx)}{L(k)} = 1.$$

Definition 3 (Space $M'(r)$) :

Let $M'(r)$, $r \in IR \setminus (-IN)$ denote the space of all distributions $f \in S'_+(IR)$ such that $f = t^{-r} D^k F$, for some $k \in INo$ and locally integrable function F, $\sup F \subset [0,\infty)$ and

$$\int_{IR} /F(t)/(t+\beta)^{-r-l-k}\, dt < \infty \text{ for } \beta > 0$$

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Definition 4 :

The Stieltjes transformation $S_r(f)(s), r \in IR \setminus (-IN)$ is complex valued function defined by

$$S_r(f(t))(s) = \int_0^\infty \frac{f(t)}{(s+t)^{r+1}} dt \tag{1}$$

$s \in \not\subset \setminus (-\infty, 0],\ 0 < t < \infty,\ r \in IR \setminus (-IN)$

Modified Stieltjes transformation introduced by Marichev [8] is

$$T_\alpha(f(x)) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(1 + \frac{x}{y}\right)^{-\alpha} \frac{1}{y} f(y) dy \quad x \in \not\subset \setminus (-\infty, 0], \quad 0 < y < \infty$$

It can be written as,

$$T_\alpha(f(x)) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{y^{\alpha-1} f(y)}{(x+y)^\alpha} dy. \tag{2}$$

By putting, $r = \alpha - 1, f(t) = y^{\alpha-1} f(y)$ in (1)
we get,

$$S_{\alpha-1}(y^{\alpha-1} f(y))(x) = \int_0^\infty \frac{y^{\alpha-1} f(y)}{(x+y)^\alpha} dy. \tag{3}$$

By equation (2) & (3)

$$T_\alpha(f)(x) = \frac{1}{\Gamma(\alpha)} S_{\alpha-1}(y^{\alpha-1} f(y))(x).$$

By interchanging x by z & α by $r+1$

$$\Gamma(r+1) T_{r+1}(f)(z) = S_r(y^r f)(z). \tag{4}$$

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Definiton 5 : Modified Stieltjes transformation T_{r+1}

Let us define the T_{r+1} - transformation $r \in IR \setminus (-IN)$. Assume that $f \in M'(r)$ and that $f = t^{-r} D^k F$ and $\int_{IR} |F(t)| (t+\beta)^{-r-1-k} < \infty$ for $\beta > 0$, holds.

Now definition is as follows

$T_{r+1}(f)$, $r \in IR \setminus (-N)$ is complex valued function defined by

$$\Gamma(r+1) T_{r+1}(f)(s) = (r+1)_k \int_0^\infty \frac{F(t)}{(s+t)^{r+1+k}} dt .$$

$r \in IR \setminus (-lN)$, $s \in \not\subset \setminus (-\infty, 0]$, $0 < t < \infty$

Definition 6 : Generalized modified Stieltjes transformation $\widetilde{T}_{r+1}$

The $\widetilde{T}_{r+1}$ - transformation of a distribution $f \in S'_+(IR)$ is complex valued function $\widetilde{T}_{r+1}(f)$ defined by

$$\Gamma(r+1) \widetilde{T}_{r+1}(f)(s) = \lim_{w \to \infty} < f(t), \eta(t)(s+t)^{-r-1} \exp(-wt) > \qquad (5)$$

$w \in IR$, $s \in \Lambda \subset (\not\subset \setminus (-\infty, 0])$, $\eta \in A(s)$.

where Λ is the set of complex number for which this limit exists and $A(s)$ is the family of all smooth functions, defined on R for which there exists $\varepsilon = \varepsilon_{\eta,s} > 0$ such that $0 \le \eta(t) \le 1$, $t \in R$, $\eta(t) = 1$ if t belongs to the ε - neighbourhood of $\overline{R}_+$, $\eta(t) = 0$ if it belongs to the complement of the 2 ε - neighbourhood of $\overline{R}_+$, where $\varepsilon > 0$ is arbitrary if $IMS \neq 0$, and $0 < 2\varepsilon <$ max Res, if for some $IMS = 0$ and $|D^p \eta(t)| \le C_p$,

$t \in IR$. If $\eta(t) \in A(s)$, $s \in (\not\subset \setminus R_-)$

$(R_- = (-\infty, 0])$, then $\eta(t)(s+t)^{-r-1} \exp(-wt) \in S(IR)$ for $w \in R_+$, $r \in R$

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INITIAL VALUE ABELIAN THEOREM

To prove this theorem we use the following definitions, lemma and propositions.

Definition 3.1 ([2]) :

Let $M'(r),\ r \in IR \setminus (-IN)$, denote the space of distributions f having supports in $l\overline{R}_+$ and admitting decomposition $f = t^{-r}(S + D^k F)$, where F is a function with support in the set $B = [\alpha, \infty), (\alpha \in IR_+)$, $F(t)t^{-r-k-1}$ is integrable and S is distribution having support in $l\overline{R}_+ \setminus \text{int}\, B$.

Now if $f = t^{-r}(S + D^p F) \in M'(r)$ the modifeid Stieltjes transformation of f defined by Lavoine & Misra [1] is

$$\Gamma(r+1)(T_{r+1}f)(s) = < S(t),\ (s+t)^{-r-1} > + \frac{\Gamma(r+q+1)}{\Gamma(r+1)} \int_{\alpha}^{\infty} F(t)(s+t)^{-r-q-1}\, dt \quad (3.1)$$

If $f \in M'(r_0)$, then we can prove that for all $r \geq r_0$, modified Stieltjes transform $\Gamma(r+1)(\widetilde{T}_{r+1}f)(s)$ given by definition of modified Stieltjes equation (5) exists and

$$\Gamma(r+1)(\widetilde{T}_{r+1}f)(s) = \Gamma(r+1)\widetilde{S}_{r+1}(s) + (r+1)_q \int_0^{\infty} \frac{F(t)}{(s+t)^{r+q+1}}\, dt,\ s \in (\not{\in} \setminus \overline{R}_-) \quad (3.2)$$

For a distribution $S \in (S'_+)$ with support in in a compact $k \subset l\overline{R}_+$, the modified Stieltjes transform is

$$\Gamma(r+1)\widetilde{S}_{r+1}(s) = \lim_{w \to 0} < S(t),\ \frac{\eta(t)\exp(-wt)}{(s+t)^{r+1}} > = < S(t),\ \frac{\xi(t)}{(s+t)^{r+1}} > \quad (3.3)$$

where, $\xi \in C^{\infty}$, $\text{supp}\xi \subset k^{3,\varepsilon}$ & $\xi(t) = 1,\ t \in k^{\varepsilon}$, k^{ε} is the ε – neighbourhood of k.

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Definition 3.2 ([2]) :

Distribution $f \in S'_+$ has the quasi-asymptotic behaviour at 0^+ related to $(1/k)^a L(1/k)$ with the limit $g \in S'_+$, $g \neq 0$, if for every $\phi \in S$

$$\lim_{k\to\infty} \frac{k^a}{L(1/k)} < f(t/k),\ \phi(t) > = < g(t),\ \phi(t) >. \tag{3.4}$$

We write in short $f \overset{q}{\sim} g$ at 0^+, related to $(1/k)^a L(1/k)$.

Proposition 3.1 :

Let f_1 & f_2, belong to S'_+ & that $f_1 = f_2$ in some neighbourhood of zero.

If $f_1 \overset{q}{\sim} g$ at 0^+, related to $(1/k)^b L(1/k)$, then $f_2 \overset{q}{\sim} g$ at 0^+, related to $(1/k)^b L(1/k)$ too.

Theorem 3.1 :

Let $f \in S'_+$ then, $f \overset{q}{\sim} g$ at 0^+ related to $(1/k)^b L(1/k)$ if and only if there exists $a \in R_+$, $a > -b-2$, such that $F = f_{a+1} * f \in L'_{loc}$, F defines a regular tempered distribution and

$F(t) \sim C f_{b+a+2}(t) L(t)$, $t \to 0^+$. we know that

$$\text{Supp } F \subset IR_+ \ \&\ |\mathbb{F}(t)| \leq (1+\|t\|)^m t^a,\ t \in R_+,\ a > m. \tag{3.5}$$

Lemma 3.1 :

Let $r-b>0$, $b+p+1>0$. Then

$$\lim_{k\to\infty} \frac{k^{-(r-b)}}{L(1/k)} (r+1)_p \int_0^\infty \frac{f_{b+p+1}(t) L(t)}{((s/k)+t)^{r+p+1}} dt = \frac{\Gamma(r-b)}{\Gamma(r+1)} s^{-(r-b)}.$$

$L(t)$ is slowly varying at 0^+, $t \in IR$ $s \in \not\subset \setminus \overline{R}_-$.

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Lemma 3.2 :

Let $G \in L'_{loc}(IR)$, supp $G \subset [0,\infty)$, $r > \alpha > -1$, $G(x) \sim x^{\alpha}$ as $x \to 0^{+}$ and

$$\int_0^{\infty} \left| \frac{G(t)}{(z+t)^{r+1}} \right| dt < \infty,\ z \in \not\subset \setminus (-\infty, 0].$$

Then,

$$z \to z^{r-\alpha}\Gamma(r+1)(T_{r+1}G)(z)$$

is a bounded analytic function in the domain $\Omega_{0,\epsilon} \cap B(0,R)$, $0 < \varepsilon < \pi/2$, $R > 0$.

$\left(\Omega_{a,\epsilon} = \left\{a + \mathrm{Re}^{i\phi}; R > 0, -\pi + \epsilon \le \phi \le \pi - \epsilon\right\}, a \ge 0 \ \&\ B(0,R) = \left\{z : |z| < R\right\}\right)$.

Theorem 3.2 : Initial value theorem

Let $f \in M'(r_0)$ and that $f \overset{q}{\sim} g$ at 0^{+} related to $(1/k)^{b} L(1/k)$ then $g = CF_{b+1}$ & for s belonging to the closed domain

$$\Omega_{\epsilon} = \left\{s \in \not\subset,\ \arg s \in [-\pi + t, -\pi - \varepsilon],\ (0 < \varepsilon < \pi/4)\right\}.$$

We have

$$\lim_{k \to \infty} \frac{(s/k)^{r-b}\Gamma(r+1)(T_{r+1}f)(s/k)}{L(1/k)} = C\frac{\Gamma(r-b)}{\Gamma(r+1)} \tag{3.6}$$

where, $r \neq 1$, $r > b$, and $r \ge r_0$. If $L = 1$ the convergence is uniform in Ω_{ϵ}.

Proof :

By definition 3.1, $f = t^{-r}(S + D^{p}F)$ where supp $f \subset B = [\alpha, \infty)$, $(\alpha \in IR_{+})$. $F(x)x^{-r-q-1}$ is integrable and S is distribution having support in the set $k \in (IR_{+} \setminus B)$.

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By proposition 3.1. $S \overset{q}{\sim} g$ at 0^+ related to $(1/k)^b L(1/k)$.

Hence, $S = t^{-r} D^p G,\ G \in L'_{loc},\ G \sim CF_{b+p+1}$ as $t \to 0^+$ $(\because$ Theorem 3.1$)$

Let k^ϵ be the ε – neighbourhood of the compact set k and let the function ξ be from C^∞ such that $\xi(t) = 1,\ t \in k,\ \xi(t) = 0,\ t \notin k^{3\varepsilon};\ |D^\alpha \xi| \le C_\alpha C(\varepsilon)$

Using Leibniz Formula, we have,

$$S(t) = \xi(t)S(t) = \xi(t)t^{-r}D^p G(t) = D^p\left(t^{-r}\xi(t)G(t)\right) - \sum_{\substack{\alpha \le p \\ \alpha \ne p}} D^\alpha\left(\xi_\alpha(t)t^{-r}G(t)\right), \qquad (3.7)$$

Where, ξ_α are functions with their support in $k^{3\alpha} \setminus k^\varepsilon([\alpha, \alpha + 3\varepsilon])$

$$\left(\because t^{\alpha-1}D^k F = D^k\left(t^{\alpha-1}F\right) - \sum_{p \le k} D^p\left(t_p^{\alpha-1}F\right)\right)$$

Let $s \in \not\subset$, $\arg s \ne \pi$. The modified Stieltjes transformation of the distribution f is

$$\Gamma(r+1)(T_{r+1}f)(s) = \langle S(t)\xi(t), \frac{\eta(t)}{(s+t)^{r+1}} \rangle + (r+1)_q \int_\alpha^\infty \frac{F(t)}{(s+t)^{r+q+1}} \qquad \text{(by 3.1)}$$

$$= (r+1)_p \left[\int_0^\alpha \frac{G(t)}{(s+t)^{r+p+1}} + \int_\alpha^{\alpha+3\varepsilon} \frac{G(t)\xi(t)}{(s+t)^{r+p+1}} dt - \sum_{\substack{\alpha \le p \\ \alpha \ne p}} \int_\alpha^{\alpha+3\varepsilon} \frac{\xi_\alpha(t)G(t)}{(s+t)^{r+p+1}} \right.$$

$$\left. + (r+1)_q \int_\alpha^\infty \frac{F(t)}{(s+t)^{r+q+1}} \right] \qquad \text{(by 3.7)}$$

Now by Lemma 3.1

$$\lim_{k \to \infty} \frac{k^{-(r-b)}}{L(1/k)} \Gamma(r+1)(T_{r+1}f)(s/k) - \frac{C\Gamma(r-b)}{\Gamma(r+1)} s^{-(r-b)}$$

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$$= \lim_{k\to\infty} \frac{k^{-(r-b)}}{L(1/k)}(r+1)_p \int_0^{\alpha} \frac{G(t) - f_{b+p+1}(t)L(t)}{\left((s/k)+t\right)^{r+p+1}} dt + \lim_{k\to\infty} \frac{k^{-(r-b)}}{L(1/k)} \int_0^{\infty} H(s/k,t)dt \quad (3.8)$$

where, $H(s/k,t), t \in IR_+$ is function with the properties. $\lim_{k\to\infty} H(s/k,t) = H(0,t)$ and $H(0,t) \in L'$.

As $r>b$

$$\lim_{k\to\infty} \frac{k^{-(r-b)}}{L(1/k)} \int_0^{\infty} H(s/k,t)dt = 0 \quad (3.9)$$

Now we have to prove that the first term of (3.8) tends to zero.

We have,

$$G(t) - Cf_{b+p+1}(t)L(t) = d(t)f_{b+p+1}(t)L(t), t \in K \quad (3.10)$$

where d is a locally integrable function and $d(t) \to 0$, when $t \to 0^+$.

Therefore we can choose k_0 in such a way that for any $\varepsilon > 0, |d(t/k)| < \varepsilon, t \in k$ and $k \geq k_0$. Then

$$\left| \frac{k^{-(r-b)}}{L(1/k)}(r+1)_p \int_0^{\alpha} \frac{G(t) - f_{b+p+1}(t)L(t)}{\left((s/k)+t\right)^{r+p+1}} dt \right|$$

$$\leq \frac{k^{-(r-b)}}{L(1/k)}(r+1)_p \int_0^{\alpha} \frac{d(t)f_{b+p+1}(t)L(t)}{\left|\left((s/k)+t\right)\right|^{r+p+1}} dt$$

$$\leq \varepsilon \frac{(r+1)_p}{L(1/k)} \int_0^{\alpha} \frac{f_{b+p+1}(u)L(u/k)}{|s+u|^{r+p+1}} du \quad (3.11)$$

$\therefore$ We proved that the first term on the right hand side of (3.8) tends to

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zero when $k \to \infty$.

$\therefore$ Equation 3.8 becomes

$$\lim_{k\to\infty} \frac{k^{-(r-b)}}{L(1/k)} \Gamma(r+1)(T_{r+1}f)(s/k) - C\frac{\Gamma(r-b)}{\Gamma(r+1)} s^{-(r-b)} = 0 \qquad \text{(by (3.9) \& (3.11))}$$

$$\therefore \lim_{k\to\infty} \frac{k^{-(r-b)}}{L(1/k)} \Gamma(r+1)(T_{r+1}f)(s/k) = C\frac{\Gamma(r-b)}{\Gamma(r+1)} s^{-(r-b)}.$$

which is required result. (by result (3.6))

For the proof of last statement of theorem. We use lemma 3.2 & Theorem 3.1 it follows that $z^{r-b}\Gamma(r+1)(T_{r+1}f)(z)$, is a bounded analytic function in $\Omega_{0,\epsilon} \cap B(0,R)$. Let $M(z) = \Gamma(r+1)(T_{r+1}f)(1/z)$, $z \in \not\subset \setminus (-\infty, 0]$.

The function $w^{r-b}M(w), w \in \Omega_{0,\epsilon} \cap \{w: |w| > 1/R\}$, $\varepsilon > 0$, $R > 0$ is analytic and bounded. The first part of theorem 3.2 implies $x^{r-b}M(x) \to A$, as $x \to \infty$, where

$$A = \frac{C\Gamma(r-b)}{\Gamma(r+1)}.$$

This implies that the same holds for $w^{r-\alpha}M(w)$ in the domain $\Omega_{a,\epsilon} \cap \{w: |w| > 1/R\}$, $a > 0$, $\varepsilon > 0$, $R > 0$.

By the Montel's theorem it follows that $w^{r-b}M(w)$ converges to A uniformly in this domain & thus $w^{r-b}M(w)$ converges uniformly to A in $\Omega_{0,\epsilon}$ as $|w| \to \infty$. This implies that

$z^{r-b}\Gamma(r+1)(T_{r+1}f)(z) \to A$ uniformly in Ω_{ϵ} as $z \to 0$.

Hence the proof of theorem.

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Lemma 3.3 :

Let $f \in M'(r)$. Then for $x \in IR_+$ and $s \in \not\subset \setminus \overline{R}_-$

$$\Gamma(r+1)(T_{r+1}f)(sx) = s(r+1)\int_x^\infty \Gamma(r+2)(T_{r+2}f)(st)dt$$

Theorem 3.3 :

If $f \in M'(r)$ and $\Gamma(r+2)(T_{r+2}f)(x) \sim Cx^{-(r-b)-1}L(x)$, $x \to 0^+$, where $b < r$,

then $\Gamma(r+1)(T_{r+1}f)(x) \sim C\dfrac{(r+1)}{(r-b)}x^{-(r-b)}L(x)$, $x \to 0^+$.

Proof :

Let $\varepsilon > 0$ be fixed. There exists $\delta \in lR_+$ such that for $0 < x < \delta$.

$$\Gamma(r+2)(T_{r+2}f)(x) = C(1+\in(x))x^{-(r-b)-1}L(x) \tag{3.12}$$

Where $\in$ is locally integrable function in the δ – neighbourhood of zero and such that

$$|C\varepsilon(x)| \le \varepsilon/8,\ 0 < x < \delta, \text{ with } M = C(r-b)^{-1} \text{ and } N = (r+1).$$

We have from Lemma 3.2.

$$\left|\frac{\Gamma(r+1)(T_{r+1}f)(x)}{NMx^{b-r}L(x)} - 1\right| \le \left|\int_x^\delta \frac{\Gamma(r+2)(T_{r+2}f)(t)dt}{Mx^{b-r}L(x)} - 1\right| + \left|\int_\delta^\infty \frac{\Gamma(r+2)(T_{r+2}f)(t)dt}{Mx^{b-r}L(x)}\right|.$$

Now there exist $\overline{\delta}$, $0 < \overline{\delta} < \delta$, such that for $0 < x < \overline{\delta}$

$$\left|\int_\delta^\infty \frac{\Gamma(r+2)(T_{r+2}f)(t)dt}{Mx^{b-r}L(x)}\right| \le \varepsilon/8.$$

Thus, for $0 < x < \overline{\delta}$ we have

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$$\left|\frac{\Gamma(r+1)(T_{r+1}f)(x)}{NMx^{b-r}L(x)}-1\right| \le \left|\int_x^{\delta}\frac{(1+\in(t))Ct^{-(r-b)-1}L(t)dt}{Mx^{b-r}L(x)}-1\right|+\varepsilon/8$$

$$\le \left|C\int_x^{\delta}\frac{t^{b-r-1}L(t)dt}{Mx^{b-r}L(x)}-1\right|+\varepsilon/8\frac{\left|\int_x^{\infty}t^{b-r-1}L(t)dt\right|}{Mx^{b-r}L(x)}+\varepsilon/8 \quad \left(\because |C\varepsilon(x)| \le \varepsilon/8\right)$$

$$\le \left|\frac{C\int_1^{\delta/x}u^{b-r-1}L(ux)du}{ML(x)}-1\right|+\varepsilon/8\frac{\left|\int_1^{\delta/x}u^{b-r-1}L(ux)du\right|}{ML(x)}+\varepsilon/8$$

$$\le \left|C\frac{\int_1^{\infty}u^{b-r-1}L(ux)du}{ML(x)}-1\right|+C\frac{\left|\int_{\delta}^{\infty}t^{b-r-1}L(t)dt\right|}{Mx^{b-r}L(x)}+\varepsilon/8\frac{\left|\int_1^{\infty}u^{b-r-1}L(ux)du\right|}{ML(x)}+\varepsilon/8<\varepsilon$$

$$\therefore \left|\frac{\Gamma(r+1)(T_{r+1}f)(x)}{NMx^{b-r}L(x)}-1\right|<\varepsilon. \qquad (3.13)$$

We made use of the relation

$$\int_1^{\infty}u^{b-r-1}L(ux)du \sim L(x)\int_1^{\infty}u^{b-r-1}du,\ x\to 0^+$$

$$\therefore \left|\frac{\Gamma(r+1)(T_{r+1}f)(x)}{NMx^{b-r}L(x)}-1\right|=0 \qquad \text{(By 3.13)}$$

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$$\therefore \frac{\Gamma(r+1)(T_{r+1}f)(x)}{NMx^{b-r}L(x)} = 1$$

$$\therefore \Gamma(r+1)(T_{r+1}f)(x) = NMx^{b-r}L(x)$$

$$\therefore \Gamma(r+1)(T_{r+1}f)(x) \sim C\frac{(r+1)}{(r-b)}x^{-(r-b)}L(x),\ x \to 0^+$$

Hence the proof.

CONCULUSION

By applying Stieltjes transforms and method given in Pilipovic [3] for quasi-asymptotic behaviour of distribution. We obtain "Initial value Abelian theorem for Distributional Modifeid Stieltjes Transformations".

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PEXIDER DIFFERENCE OPERATOR IN L^P SPACES

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ABSTRACT

Assume that (G, +) is a complete measurable group with an finite measure λ and E is a Banach space. For $f, g, h : G \to E$ define the Pexider difference operator by $P(f, g, h)(x, y) := f(x+y)-g(x)-(h)\,y,\ x, y \in G$.

In this paper we present some results about such Pexider operator. We prove theorem about the Hyers-Ulam-Rassias stability of Pexider operator. Moreover, theorems concerning continuity and the existence of continuous inverse operator to Pexider operator are established.

INTRODUCTION

Let E be a Banach space and let $(G,+,\Sigma,\lambda)$ be a semigroup with zero with a complete note identicaly zero measure λ and such that $\lambda(G) < +\infty$.

For any functions $f,g,h : G \to E$ we define Cauchy and Pexider operators respectively as follows

$$CF(x,y) : f(x+y)-f(x)-f(y),$$

$$P(f,g,h)(x,y) : = f(x+y)-g(x)-h(y).$$

Now we prove

Lemma 1.1 For every $f,g,h : G \to E$ we have

$$Cf(x,y) = P(f,g,h)(x,y)-P(f,g,h)(0,y)-P(f,g,h)(x,0)-h(0)-g(0), \qquad (1)$$

$$Cg(x,y) = P(f,g,h)(x,y)-P(f,g,h)(x+y,0)+P(f,g,h)(y,0)-P(f,g,h)(0,y)-g(0), \qquad (2)$$

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$$Ch(x,y) = P(f,g,h)(x,y) - P(f,g,h)(0,x+y) + P(f,g,h)(0,x) - P(f,g,h)(x,0) - h(0). \tag{3}$$

Proof. We get for $x, y \in G$,

$$Cf(x,y) = f(x+y) - f(x) - f(y) = f(x+y) - g(x) - h(y) + g(x) + h(0)$$

$$- f(x) + h(y) + g(0) - f(y) - g(0) - h(0)$$

$$= P(f,g,h)(x,y) - P(f,g,h)(x,0) - P(f,g,h)(0,y) - g(0) - h(0).$$

Analogously, we obtain

$$Cg(x,y) = g(x+y) - g(x) - g(y) = g(x+y) + h(0) - f(x+y)$$

$$+ f(x+y) - g(x) - h(y) + h(y) + g(0) - f(y) + f(y) - g(y) - h(0)$$

$$= P(f,g,h)(x,y) - P(f,g,h)(x+y,0) + P(f,g,h)(y,0) - P(f,g,h)(0,y) - g(0).$$

The proof of (3) is quite similar.

Now we establish some denotation and general hypothesis. We define

${}_yA = \{t \in G; y+t \in A\}$, $A_y = \{t \in G; t+y \in A\}$ for $A \in \Sigma$.

We assume :

(p) $A_y \in \Sigma$ and $\lambda(A_y) = \lambda(A)$ for all $A \in \Sigma$ and all $A \in G$,

(l) ${}_yA \in \Sigma$ and $\lambda({}_yA) = \lambda(A)$ for all $A \in \Sigma$ and all $y \in G$.

The following denotations will be used further on.

For a complete measure space (X, Σ, ν) and $1 \le p \le +\infty$ we denote (see [1], [2], [5]) $L_\nu^p(X,E) := \{f : X \to E;\ f$ is Σ measurable and $|f|_p < +\infty\}$,

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where $|f|_p := \left(f_x \|f(x)\|^p dv(x)\right)^{1/p}$ for $1 \le p < +\infty$, $|f|_\infty := \text{esssup}_{t \in X} \|f(t)\|$.

Of course, function f and g are equal if they are equal almost everywhere with respect to the measure v.

If $f: G \to E$ and $y \in G$, then we put $f^y(x) := f(y+x), f_y(x) := f(x+y)$. For $f: G \times G \to E$, we denote $f^0(x) := f(0,x), f_0(x) := f(x,0)$, $f^+(x,y) := f(x+y,0)$, $f_+(x,y) := f(0,x+y)$.

We will apply the following

Lemma 1.2 ([6] Let (G, Σ, λ) be a complete measure space such that $0 < \lambda(G) < +\infty$ and let $f \in L^1_\lambda(G,E)$. If (p) is satisfied, then for $y \in G$ $f_y \in L^1_\lambda(G,E)$ and $\int_G f_y(t)$ $d\lambda(t) = \int_G f(t) d\lambda(t)$. Moreover, if (I) is satisfied, then for $y \in G f^y \in L^1_\lambda(G,E)$ and $\int_G f^y(t) d\lambda(t) = \int_G f(t) d\lambda(t)$.

STABILITY OF THE PEXIDER EQUATION

In this section we are going to give the positive answer to the problem of Hyers-Ulam-Rassias stability (see also [3], [4]) of the Pexider equation :

$$f(x+y) = g(x) + h(y), \; x, y \in G.$$

The problem can be formulated as follows.

Let $f, g, h: G \to E$ be such that $P(f,g,h) \in L^p_{\lambda \times \lambda}(G \times G, E)$. Find Constant $M \ge 0$ and an additive mapping $A: G \to E$ such that the following conditions are satisfied

$$|f - A|_p \le M \|P(f,g,h)\|_p,$$

$$|g - A|_p \le M \|P(f,g,h)\|_p,$$

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$$|h-A|_p \le M\|P(f,g,h)\|_p.$$

Note that $\|P(f,g,h)\|_p := \left[\int\limits_{G\times G}\|P(f,g,h)(x,y)\|^p d\lambda(x)d\lambda(y)\right]^{1/p}$.

For the Cauchy equation this problem has been solved in [6] in the case $\lambda(G)<+\infty$ and in the case $\lambda(G)=+\infty$ in [7].

By μ we understand the completion of the product of the measure λ times itself.

The following theorem, which we will apply latter on, can be found in [6].

Theorem 2.1 Let G be a semigroup and let (G,Σ,λ) be a complete measure space satisfying at least one of the conditions (p), (l). We assume that the mapping $s: G\times G \ni (x,y)\to(x,x+y)\in G\times G$ is measurable. Let E be a Banach space and let $f: G\to E$ be such that $Cf\in L^p_\mu(G\times G,E)$ for a certain $l\le p\le+\infty$. If $0<\lambda(G)<+\infty$, then there exists function $A: G\to E$ such that

(i) $CA=0$

(ii) $f-A\in L^p_\lambda(G,E)$,

(iii) $|f-A|_p \le \lambda(G)^{-1/p}\|Cf\|_p$,

(iv) for every $A^*: G\to E$ satisfying the conditions $CA^*=0$, and $f-A^*\in L^p_\lambda(G,E)$, we have $A=A^*$.

The next result is the following

Theorem 2.2 Let the assumptions of Theorem 2.1 be satisfied. Moreover, Let $f,g,h: G\to E$ be such that $P(f,g,h), P(f,g,h)^+, P(f,g,h)_+ \in L^p_\mu(G\times G,E)$,

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$P(f,g,h)^0, P(f,g,h)_0 \in L^p_\lambda(G,E)$ for a fixed $1 \le p \le +\infty$. If $0 < \lambda(G) < +\infty$, then there exists $A : G \to E$ such that

(v) $CA = 0$

(vi) $f - A, g - A, h - A \in L^p_\lambda(G,E)$,

(vii) a) $|f - A|_p \le \lambda(G)^{-1/p} \|Cf\|_p$,

b) $|g - A|_p \le \lambda(G)^{-1/p} \|Cg\|_p$,

c) $|h - A|_p \le \lambda(G)^{-1/p} \|Ch\|_p$,

(viii) a) if $B : G \to E$, $CB = 0$, $f - B \in L^p_\lambda(G,E)$, then $B = A$,

b) if $D : G \to E$, $CD = 0$, $g - D \in L^p_\lambda(G,E)$, then $D = A$,

c) if $H : G \to E$, $CH = 0$, $h - H \in L^p_\lambda(G,E)$, then $H = A$.

Proof. At first we quess, in view on Lemma 1.1, that $Cf, Cg, Ch \in L^p_\mu(G \times G, E)$. Therefore, making use of Theorem 2.1, we obtain that there exist A, A_1, $A_2 : G \to E$ such that

$$CA = CA_1 = CA_2 = 0, \tag{4}$$

$$f - A, g - A_1, h - A_2 \in L^p_\lambda(G,E),$$

$$|f - A|_p \le \lambda(G)^{-1/p} \|Cf\|_p,$$

$$|g - A_1|_p \le \lambda(G)^{-1/p} \|Cg\|_p, \tag{5}$$

$$|h - A_2|_p \le \lambda(G)^{-1/p} \|Ch\|_p.$$

We shall prove that $A = A_1 = A_2$ (with respect to the measure λ).

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Note that we have

$$g(x)-A(x)=-P(f,g,h)(x,0)-h(0)+f(x)-A(x),$$

$$h(x)-A(x)=-P(f,g,h)(x,0)-g(0)+f(x)-A(x),$$

Whence, using our assumptions, we get

$$g-A,h-A\in L_\lambda^p(G,E). \tag{6}$$

From Theorem 2.1 we have

if $B:G\to E$ satisfies $CB=0$, $g-B\in L_\lambda^p(G,E)$, then $B=A_1$,

if $D:G\to E$ satisfies $CD=0$, $h-D\in L_\lambda^p(G,E)$, then $D=A_2$,

Hence, taking $B=A$ and $D=A$, in view of (4) and (6) we get $A=A_1=A_2$.

The conditions (vii) and (viii) we obtain now directly from (5) and (7) (substitiuting $A=A_1=A_2$). This completes the proof.

Corollay 2.1 Under The assumption of Theorem 2.2 istead of inequalities (vii) we have

(ix) a) $$|f-A|_p\le\lambda(G)^{-1/p}\|P(f,g,h)\|_p+\left|P(f,g,h)^0\right|_p$$

$$+\left|P(f,g,h)_0\right|_p+\lambda(G)^{1/p}\left(\|g(0)\|+\|h(0)\|\right),$$

b) $$|g-A|_p\le\lambda(G)^{-1/p}\|P(f,g,h)\|_p+\left|P(f,g,h)^+\right|_p+$$

$$\left|P(f,g,h)^0\right|_p+\left|P(f,g,h)_0\right|_p+\lambda(G)^{1/p}\|g(0)\|,$$

c) $$|h-A|_p\le\lambda(G)^{-1/p}\|P(f,g,h)\|_p+\left|P(f,g,h)_+\right|_p+$$

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$$\left|P(f,g,h)^0\right|_p + \left|P(f,g,h)_0\right|_p + \lambda(G)^{1/p}\|h(0)\|.$$

Proof. We will prove the condition a). The inequalities b) and c) one can verify analogously. In virtue of Lemma 1.1 we have $Cf(x,y) = P(f,g,h)(x,y) - P(f,g,h)^0(x,y) - P(f,g,h)_0(x,y) - g(0) - h(0)$. Hence we get

$$\|Cf\|_p \le \|P(f,g,h)\|_p + \left\|P(f,g,h)^0\right\|_p + \left\|P(f,g,h)_0\right\|_p + \|g(0)\|_p + \|h(0)\|_p \le \|P(f,g,h)\|_p +$$

$$\lambda(G)^{1/p}\left|P(f,g,h)^0\right|_p + \lambda(G)^{1/p}\left|P(f,g,h)_0\right|_p + \lambda(G)^{2/p}\left[\|g(0)\| + \|h(0)\|\right].$$

Consequely, applying the inequality (vii) a), we obtain the condtion (ix) a), which completes the proof.

PEXIDER LINEAR OPERATOR

Recall the following theorem contained in [6].

Theorem 3.1 Let G be a non-empty set with a binary operation +. Let (G,Σ,λ) be a measure space satifying one of the conditions (p) or (l). If $0 < \lambda(G) < +\infty$, $f \in L^p_\lambda$ for a certain $1 \le p \le +\infty$ and the mapping $G^2 \ni (x,y) \to f(x+y)$ is measurable in the product space, than

(x) $Cf \in L^p_\mu(G \times G, E)$,

(xi) $|f|_p \le \lambda(G)^{-1/p}\|Cf\|_p$,

(xii) $\|Cf\|_p \le 3\lambda(G)^{1/p}|f|_p$.

Now we prove

Theorem 3.2 Let the assumptions of Theorem 3.1 be satisfied. If $f,g,h \in L^p_\lambda$ for a certain $1 \le p \le +\infty$, $0 \in G$ and the mapping $G^2 \ni (x,y) \to f(x+y)$,

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$G^2 \ni (x,y) \to g(x+y)$, $G^2 \ni (x,y) \to h(x+y)$ are measurable in the product space, then

(xiii) Operator : $(f,g,h) \to P(f,g,h)$ is linear,

(xiv) $P(f,g,h) L_\mu^p(G \times G, E)$,

(xv) If $h(0) = g(0) = 0$, then

$$|f|_p + |g|_p + |h|_p \le 3\lambda(G)^{-1/p}\left(\left\|P(f,g,h)\right\|_p + \left\|P(f,g,h)^+\right\|_p + \left\|P(f,g,h)_+\right\|_p\right) +$$

$$3\left(\left|P(f,g,h)^0\right|_p + \left|P(f,g,h)_0\right|_p\right),$$

(xvi) If $h(0) = g(0) = 0$, then $\lambda(G)^{-1/p}\left(\left\|P(f,g,h)\right\|_p + \left\|P(f,g,h)^+\right\|_p + \left\|P(f,g,h)_+\right\|_p\right)$

$$+\left|P(f,g,h)^0\right|_p + \left|P(f,g,h)_0\right|_p \le 5\left(|f|_p + |g|_p + |h|_p\right).$$

Proof. If $f,g,h,k,l,m \in L_\lambda^p$, and $s \in K$ (set of real or complex number), then we get easily

$$P(f+k, g+l, h+m) = P(f,g,h) + P(k,l,m),$$

$$P(sf, sg, sh) = sP(f,g,h)$$

so P is linear operator.

Let $l_r : G \times G \to E$, $r = 1,2,3$ be defined by : $l_1(x,y) := f(x+y)$, $l_2(x,y) : g(x)$, $l_3(x,y) : h(y)$. Now we have by the Fubini-Tonelli theorem

$$\|l_2\|_p^p = \int_{G\times G} \|l_2(x,y)\|^p d\mu(x,y) = \int_{G\times G} \|g(x)\|^p d\mu(x,y) =$$

$$\int_G d\lambda(y) \int_G \|g(x)\|^p d\lambda(x) = \lambda(G)|g|_p^p < +\infty$$

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and hence $l_2 \in L_\mu^p$. By the same arguments we get $l_3 \in L_\mu^p$ and $\|l_3\|_p^p = \lambda(G)|h|_p^p$.

Similarly in view of lemma 1.2 we obtain

$$|f|_p^p = \int_G \|f(x)\|^p d\lambda(x) = \int_G \|f(x+y)\|^p d\lambda(x) = \lambda(G)^{-1} \int_G d\lambda(y) \int_G \|l_1(x,y)\|^p d\lambda(x) =$$

$$\lambda(G)^{-1} \int\limits_{G\times G} \|l_1(x,y)\|^p d\lambda(x,y) = \lambda(G)^{-1} \|l_1\|_p^p,$$

whence $|f|_p = \lambda(G)^{-1/p} \|l_1\|_p$. Therefore $P(f,g,h) = l_1 - l_2 - l_3 \in L_\mu^p$ and (xiv) has been proved.

Consequently we have also proved that

$$\|P(f,g,h)\|_p \le \|l_1\|_p + \|l_2\|_p + \|l_3\|_p = \lambda(G)^{1/p} \left(|f|_p + |g|_p + |h|_p\right). \tag{8}$$

Now we shall prove (xv). From Theorem 3.1 we have the inequalities

$$|f|_p \le \lambda(G)^{-1/p} \|Cf\|_p, |g|_p \le \lambda(G)^{-1/p} \|Cg\|_p, |h|_p \le \lambda(G)^{-1/p} \|Ch\|_p.$$

Therefore, by Lemma 1.1, we obtain successively

$$|f|_p + |g|_p + |h|_p \le \lambda(G)^{-1/p} \left(\|Cf\|_p + \|Cg\|_p + \|Ch\|_p\right) \le \lambda(G)^{-1/p} \left(\left\| P(f,g,h) - P(f,g,h)^0 - P(f,g,h)_0 \right\|_p \right.$$

$$+ \left\| P(f,g,h) - P(f,g,h)^+ - P(f,g,h)^0 + P(f,g,h)_0 \right\|_p + \left\| P(f,g,h) - P(f,g,h)_+ - \right.$$

$$\left. - P(f,g,h)_0 + P(f,g,h)^0 \right\|_p \Big) \le 3\lambda(G)^{-1/p} \left(\|P(f,g,h)\|_p + \left\|P(f,g,h)^+\right\|_p + \left\|P(f,g,h)_+\right\|_p \right) +$$

$$+ 3\left(\left|P(f,g,h)^0\right|_p + \left|P(f,g,h)_0\right|_p \right).$$

Similarly one can easily check that

$$\left|P(f,g,h)^0\right|_p \le |f|_p + |h|_p, \left|P(f,g,h)_0\right|_p \le |f|_p + |g|_p, \left\|P(f,g,h)^+\right\|_p \le \lambda(G)^{1/p}$$

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$\left(|f|_p + |g|_p\right)$, $\left\|P(f,g,h)_+\right\|_p \le \lambda(G)^{1/p}\left(|f|_p + |h|_p\right)$.

Adding the above inequalities and the inequality (8), we get (xvi), which proves the theorem.

Let's introduce some definitions.

$$DP := \left\{(f,g,h) \in L_\lambda^p \times L_\lambda^p \times L_\lambda^p; P(f,g,h)^+ \in L_\mu^p, P(f,g,h)_+ \in L_\mu^p, P(f,g,h) \in L_\mu^p,\right.$$

$$\left. h(0) = 0, g(0) = 0\right\}$$

In DP we define the norm : $\|(f,g,h)\| := |f|_p + |g|_p + |h|_p, (f,g,h) \in DP,$

and in L_μ^p the following one : $\|F\| := \lambda(G)^{-1/p}\left(\|F\|_p + \left\|F^+\right\|_p F_+\right) +$

$$\left|F^0\right|_p + |F_0|_p, F \in L_\mu^p.$$

Remark 3.1 The set DP is a linear supspace of $L_\lambda^p(G,E) \times L_\lambda^p(G,E) \times L_\lambda^p(G,E)$.

Remark 3.2 The following equality $DP = \{(f,g,h) \in L_\lambda^p \times L_\lambda^p \times L_\lambda^p; (x,y) \to f(x+y), (x,y) \to g(x+y), (x,y) \to h(x+y)$ are measurable in the product space$\}$ is the immadiate consequence of the properties of measurable functions and our Theorem 3.2.

Now we are in the position to present.

Theorem 3.3 Let the assumptions of Theorem 3.2 be satisfied and $g(0) = h(0) = 0$. Then the linear operator $P : DP \ni (f,g,h) \to P(f,g,h) \in L_\lambda^p(G \times G, E)$ is continuous and continuously invertible. Moreover, the inverse operator (defined for all $m \in P(DP)$ has the form :

$$P^{-1}m(.) = \left(-\lambda(G)^{-1}\int_G \left[m(x,.) - m_0(x) - m^0(.)\right]d\lambda(x), -\lambda(G)^{-1}\int_G \left[m(x,\cdot) - m^0(\cdot)\right.\right.$$

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$-m^0(.)+m_0(.)\big]d\lambda(x), -\lambda(G)^{-1}\int_G\big[m(x,.)-m_+(x,.)-m_0(x)+m^0(x)\big]d\lambda(x)\Big)$ if (p) occurs and

$$P^{-1}m(.)=\Big(-\lambda(G)^{-1}\int_G\big[m(.,y)-m_0(.)-m^0(y)\big]d\lambda(y), -\lambda(G)^{-1}\int_G\big[m(.,y)-m^+$$

$$(.,y)-m^0(y)+m_0(y)\big]d\lambda(y), -\lambda(G)^{-1}\int_G\big[m(.,y)-m_+(.,y)-m_0(.)+m^0(.)\big]d\lambda(y)\Big)$$ if (l) occurs.

Proof. The continuity of P follows from Theorem 3.2 point (xvi) and Remarks 3.2. The existence of inverse operator P^{-1} are the consequence of Remark 3.2, Theorem 3.2 point (xv) and Theorem 3, *p.* 43 from [8].

Now we establish the form of p^{-1}. Let us assume the condition (p) (in the case (1) the proof runs quite similarly). Note that $Cf, Cg, Ch \in L^1_\lambda(G,E)$ and in view of **Lemma 1.2** we have $f(y)=-\lambda(G)^{-1}\int_G Cf(x,y)d\lambda(x)$, $g(y)=-\lambda(G)^{-1}\int_G Cg(x,y)d\lambda(x)$, $h(y)=-\lambda(G)^{-1}\int_G Ch(x,y)d\lambda(x)$.

Making use of Lemma1.1 we obtain equalities

$$f(y)=-\lambda(G)^{-1}\int_G\big(P(f,g,h)(x,y)-P(f,g,h)(x,0)-P(f,g,h)(0,y)\big)d\lambda(x),$$

$$g(y)=-\lambda(G)^{-1}\int_G\big(P(f,g,h)(x,y)-P(f,g,h)(x+y,0)$$

$$+P(f,g,h)(y,0)-P(f,g,h)(0,y)\big)d\lambda(x),$$

$$h(y)=-\lambda(G)^{-1}\int_G\big(P(f,g,h)(x,y)-P(f,g,h)(0,x+y)+$$

$$P(f,g,h)(0,x)-P(f,g,h)(x,0)\big)d\lambda(x).$$

Hence we obtain the form of P^{-1}. This completes the proof.

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